1 Differential Equation Investigations using Customizable Mathlets

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Abstract. The author has developed some platform independent, freely available, interactive programs (mathlets) for investigating the graphs of first and second order differential equations, as well as the graphs of functions, parametric curves and data points. With these programs one can dynamically change parameters or initial conditions, and see the results immediately in multiple views (phase or time plots). This paper focuses on activities that can be used in a first course in differential equations. Mathlets which can be used to investigate mass-spring systems, pendulum systems, and population growth models are presented. Concepts from differential equations which are addressed include linear and nonlinear beats, bifurcation via the trace-determinant plane, and the Poincare map for periodic first-order equations.

1.1 Introduction: The mass-spring system

The author has developed some platform independent, freely available, interactive programs (mathlets) for investigating the graphs of first and second order differential equations, as well as the graphs of functions, parametric curves and data points. With these programs one can dynamically change parameters or initial conditions, and see the results immediately in multiple views simultaneously (phase and time plots). If a computer algebra system is available, for example Maple, the results can then be imported into Maple for further refinement there, and for report writing.

When a mathlet initially opens up it is targeted toward the investigation of a particular topic in a first course in differential equations. However, because the programs are customizable, they can then be changed to investigate a related (or unrelated) topic, and the changes can be saved. For example, one applet has been designed by the author to study the standard damped mass-spring differential equation

$$mx'' + cx' + kx = 0. \tag{1}$$

Second-order equations must be entered as first-order systems, so this equation is entered in the form $dx/dt = y$, $dy/dt = -\frac{c}{m}y - \frac{k}{m}x$. The mathlet opens up with a phase plot and two time plots (see Figure 1). One can set multiple initial conditions by double clicking (or pressing the Keep IC button) in any of the three views. Clicking and dragging allows the user to immediately see the changes in the solution curve as the initial condition is varied. Any of the three parameters m, c, k can be adjusted by typing in new values or by using the slider; changes are seen immediately and smoothly as in an animation. Thus one can observe the transition from no damping, to underdamping, to overdamping dynamically. One can trace all three curves simultaneously (similar to a graphing calculator, but with three views) to see how the three views are related. Right-clicking and dragging a rectangle creates a zoom box.

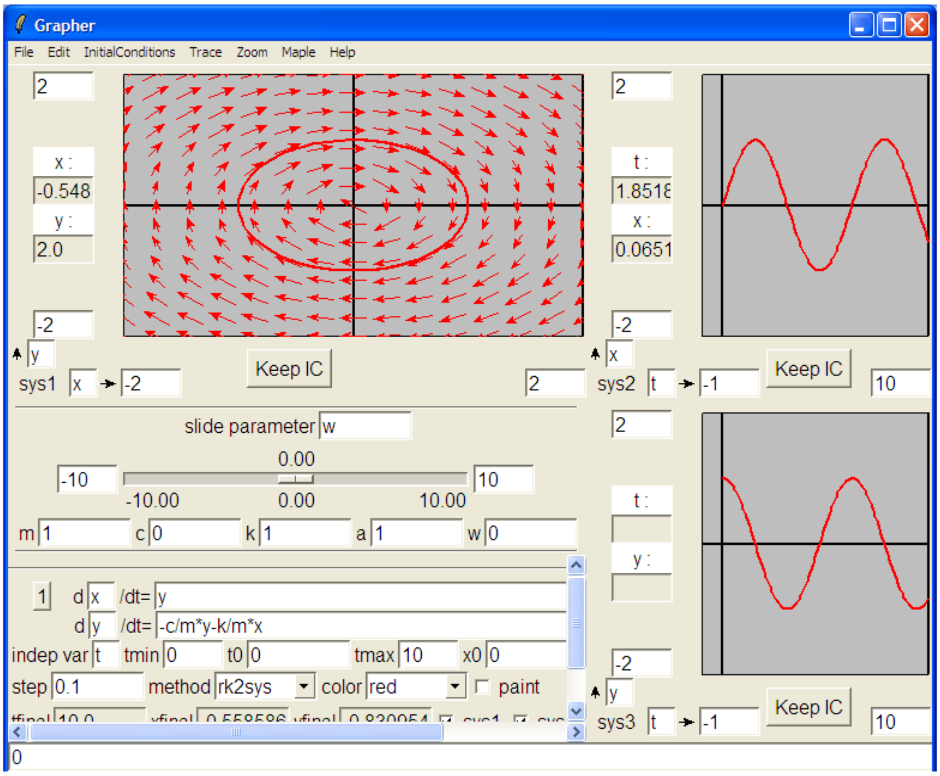

Figure 1: Mass-spring system $mx'' + cx' + kx = 0$

In order change the applet to one that represents a mass-spring system with a driving force

$$mx'' + cx' + kx = a\sin(wt) \tag{2}$$

one simply adds the term $+a\sin(wt)$ to the second differential equation to get the system $dx/dt = y$, $dy/dt = -\frac{c}{m}y - \frac{k}{m}x + a\sin(wt)$. See Figure 2. To investigate the phenomena of beats and resonance, it is necessary to adjust some of the settings in the function window (the value of $tmax$ is changed to 100, and the x and y initial conditions are set to 0 and 0) and the range settings in the graph windows (from -2 and 2 to -100 and 100 for x and y, and from 10 to 100 for t). The damping parameter c must be set equal to 0. Now when the parameter w is slowly varied between about 0.5 and 1.5, one sees first the emergence of beats (at around $w = 0.7$), then one observes the beats getting longer and larger, eventually turning into resonance at $w = 1$ (and back again into beats as w increases beyond 1). When students start to play around with interactive software, and see things like the emergence of beats, they often start to ask questions. They may ask why beats first appear at about $w = 0.7$; this is a good topic for undergraduate investigation/research (for a discussion of how to calculate when beats emerge, see [3]).

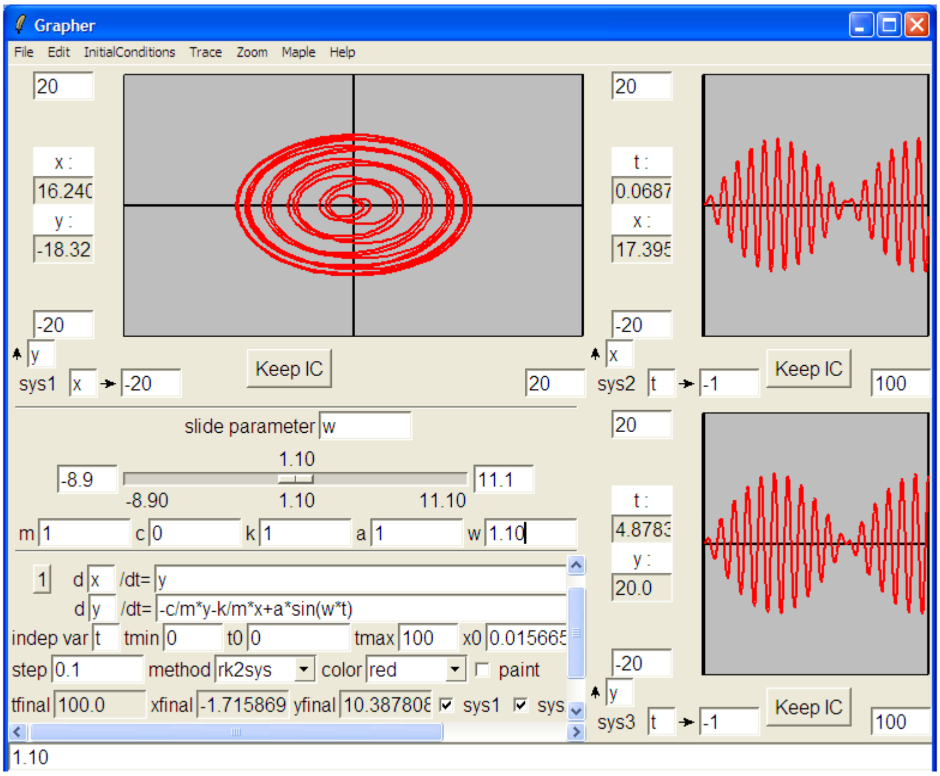

Figure 2: Driven mass-spring system $mx'' + cx' + kx = a\sin(wt)$

1.2 Follow-up: The driven pendulum

An excellent follow-up project (or demonstration by the teacher) is to investigate similar behavior for a damped, rigid pendulum

$$mx'' + cx' + k\sin(x) = a\sin(wt) \tag{3}$$

(for a pendulum the parameters m, c, k do not have precisely the same interpretation as for a mass-spring system, but to keep the analogy transparent we keep the same parameter names). The equation system now becomes $x' = y$, $y' = -\frac{c}{m}y - \frac{k}{m}\sin x + a\sin(wt)$ (one has only to change the second equation for the system in Figure 2). By choosing $a = 0$ one has an applet for a free pendulum, and for $a \neq 0$ we get a driven pendulum. One interesting activity is to continuously vary (using a slider) the forcing frequency w of a driven, undamped pendulum, given by $y" + \sin(y) = a\sin(wt)$, and observe the result in three views at once (phase plot and time plots). This time the results are a bit more complex than for the mass-spring system.

Whether or not beats are observed as w varies is dependent on the value of the amplitude a of the forcing term. For $a = 0.05$ we do see beats emerge, but something new happens also. For a mass-spring system, the beats have the same shape for w less than the resonant value of $w = 1$ as they do for w greater than $w = 1$. For the pendulum, however, the beats for w less than the"resonant" value are more rounded than those of the corresponding mass-spring system, and for w greater than the "resonant" value they are nearly diamond shaped. See Figure 3 (Figure 3 was created using the "Copy Maple commands" feature of the mathlet, and then pasting the commands into Maple). By resonant value in this case

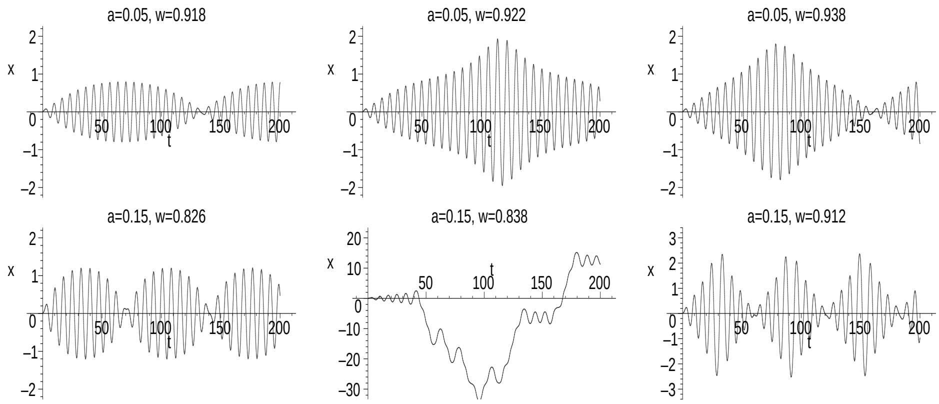

Figure 3: Beats in the pendulum equation $x'' + \sin(x) = a\sin(wt)$

we mean the value of w that results in a maximal beat period; this value is no longer $w = 1$ but somewhere around $w = 0.92$.

For a slightly larger value of a, chaos breaks loose. Choosing $a = 0.15$, and again varying w, we see the emergence of beats, with a transition to chaotic behavior, and a return to beats. The shape of the beats is similar to when $a = 0.05$. See Figure 3. Finally, if we move a up to $a = 1.0$, we find no beats at all in the range $0.5 < w < 1.5$ (primarily chaos).

While all of this behavior can be observed with any software capable of graphing differential equations, when one is using an interactive/dynamic tool, these behaviors leap out at the user as she/he plays with the software. It is fair to say that the author would not have noticed most of what is described above without such a tool. Also, the state of the mathlet can be saved, and returned to later. Thus what starts out as a single mathlet, can be customized (by the teacher, or by the student) to several tools for investigating multiple phenomena.

1.3 Bifurcation in the undriven pendulum

Another activity is to continuously vary the amount of damping for a free pendulum $my" + cy\prime + k\sin(y) = 0$, and try to identify bifurcation values. For this we use a different mathlet; in addition to a phase plot of the differential equation, there is a trace-determinant plane graph (for details of the trace-determinant plane see [2]). The fixed points are plotted in the phase plane, and the corresponding stability of each fixed point is plotted in the trace-determinant plane (the green point in the trace-determinant plane gives the stability of the green point in the phase plane, and similarly for the yellow points). See Figure 4.

After an initial exploration of various initial conditions, with a fixed value of the damping coefficient, a complete phase portrait can be generated. Then the damping coefficient can be increased continuously; one observes the approximate point at which the stable fixed points change from spiral sinks to sinks. In the window that represent the phase portrait of the differential equation, one sees the solution curve stop spiraling/oscillating; at the same time one sees in the trace-determinant plane that the yellow point is crossing the parabola $determinant = \frac{1}{4}trace^2$ (that is, the trace-determinant point is crossing into the sink region from the spiral sink region). This is the situation show in Figure 4.

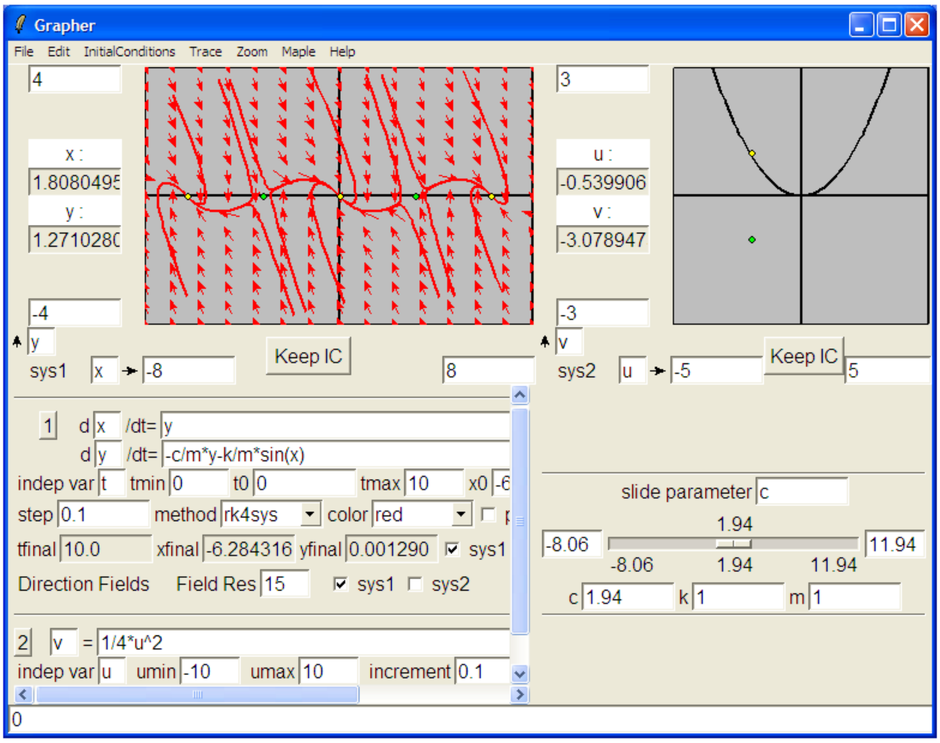

Figure 4: Phase portrait and trace-determinant for the pendulum equation $mx'' + cx' + k\sin(x) = 0$

1.4 Population growth and the Poincare map

The last activity shows how a relatively advanced concept, the Poincare map, can be made accessible to undergraduates through the use of mathlets. A model which is often investigated in a first course in differential equations is the logistic population growth equation $x' = ax(1 - x/b)$. Here, x is the population size, a represents the growth rate when the population is small, and b represents the maximum sustainable population (and hence the limit of the population as $t \to \infty$). If we apply this model to a population of fish, then to model the effects of fishing we could add a term to represent the number of fish, c, removed per unit time (say, per year). The new equation would be

$$x' = ax(1 - x/b) - c \tag{4}$$

Finally, if we want to represent seasonal fishing, we could multiply c by a seasonality term $1 - \cos(2\pi t)$ (minimum fishing when $t = 0$ or $t = 1$, and maximum fishing when $t = 1/2$). The new equation is

$$x' = ax(1 - x/b) - c(1 - \cos(2\pi t)). \tag{5}$$

Equation 4 can be studied by students using the tools of fixed points and stability analysis; since it is quadratic in x, there are either 0, 1, or 2 fixed point solutions. A mathlet designed to study this equation is shown in Figure 5. In addition to the window

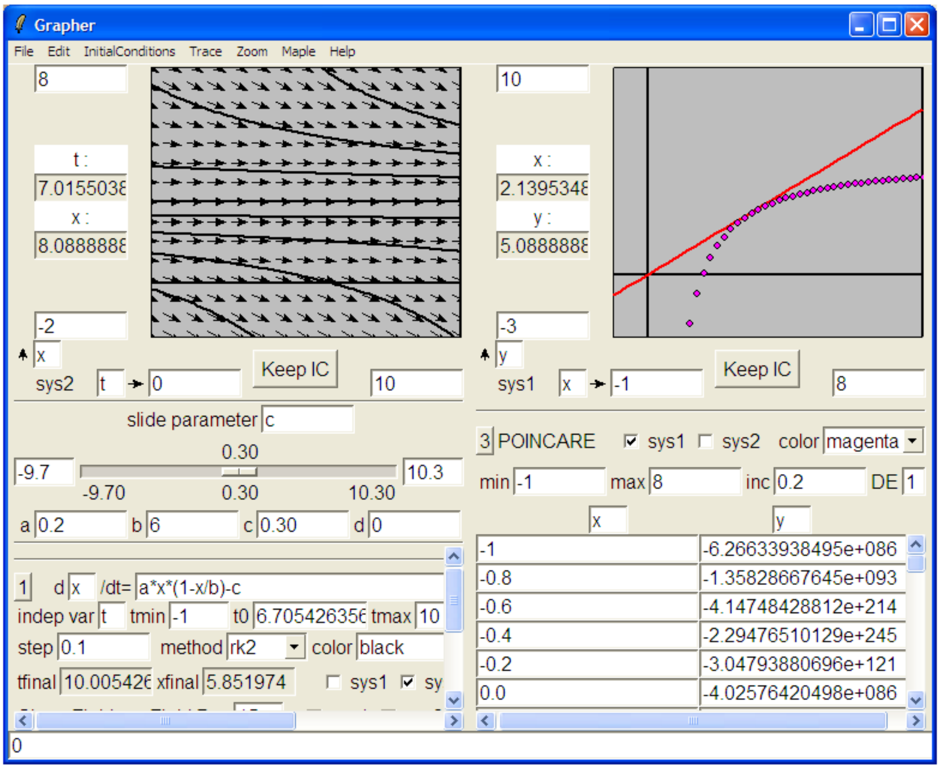

Figure 5: Poincare map for for $x' = ax(1 - x/b) - c$ at the bifurcation point

which contains the graph of the differential equation, there is a window which shows a rough Poincare map. The Poincare map takes an initial value from a differential equation $x_0 = x(t_0)$ to its value $x(t_0 + t_{period})$, t_{period} units later, where for Equation 4, t_{period} is equal to 10. Thus, if an initial point x_0 gets mapped to the same point x_0 at $t = 10$, we know that $x = x_0$ is a constant solution (fixed point). Hence whenever the Poincare map crosses the line $y = x$, the differential equation has a fixed point.

Now it is possible to estimate the value of c for which there is exactly one fixed point. This is the bifurcation value of c for which the number of fixed points changes from 2 to 0. One simply uses the slider to change the value of c until the Poincare graph becomes tangent to the line $y = x$ (as seen in Figure 5).

Equation 5 does not have any fixed points, as it is nonautonomous. It is, however, periodic in t with period 1. In such a situation, the Poincare map can be used to look for periodic solutions, rather than fixed points. As with fixed points, the intersections of the Poincare map with the line $y = x$ represent the existence of periodic solutions. Such solutions are hard to find without the use of the Poincare map, but after coming to understand how the Poincare map picks out the fixed points of an autonomous equation, it is conceptually fairly easy to see how it can be used to pick out the periodic solutions of a periodic differential equation.

The search for periodic solutions to various polynomial and polynomial-like differential equations with periodic coefficients is an active area of research (see [1]). With the use of interactive/dynamic software, undergraduates gain access to a topic which is both closely related to standard undergraduate topics, and to current mathematical research.

[1] Benardette, D., V. Noonburg and B. Pollina. *Periodic Solutions and Bifurcations of a Periodically Harvested Logistic Equation.* Amer. Math. Monthly, accepted, 2006.

[2] Blanchard, P., R. Devaney, and G. Hall. 2002. *Differential Equations, 2e.* Brooks/Cole.

[3] Decker, R., and V. Noonburg. In preparation. *Differential Equations for Scientists and Engineers.*