Note Title 10/12/2009

(1) $xy'' + y' - 4y = 0 \rightarrow y'' + \frac{1}{x}y' - \frac{4}{x}y = 0$

$p = \frac{1}{x} \quad p_0 = 1 \qquad q = -\frac{4}{x} \quad q_0 = \lim_{x \to 0} -4x = 0$

INDICIAL EQ $\quad r(r-1) + r = 0 \rightarrow r^2 = 0 \rightarrow r = 0$

$$y = x^r \sum_{n=0} a_n x^n = \sum_{n=0} a_n x^{n+r} \left(k = n \right) = \sum_{k=0}^{\infty} a_k x^{k+r}$$

$$y' = \sum_{n=0} a_n (n+r) x^{n+r-1} \left(\begin{matrix} k = n-1 & \text{START} \\ n = k+1 & k = -1 \end{matrix} \right) = \sum_{k=-1} a_{k+1} (k+1+r) x^{k+r}$$

$$y'' = \sum_{n=0} a_n (n+r)(n+r-1) x^{n+r-2}$$

$$xy'' = \sum_{n=0} a_n (n+r)(n+r-1) x^{n+r-1} \left(\begin{matrix} k = n-1 & \text{START} \\ n = k+1 & k = -1 \end{matrix} \right)$$

$$= \sum_{k=-1} a_{k+1} (k+1+r)(k+r) x^{k+r}$$

PLUG INTO (1)

$k = -1$: $\quad a_0 (r)(-1+r) + a_0 (r) = 0$

$a_0 (r^2) = 0 \quad$ SO $\quad r = 0, \; a_0 = $ CONSTANT

INDICIAL EQ AGAIN

SET $r = 0$ FROM HERE ON

$k = 0, 1, 2, \ldots : \quad a_{k+1}(k+1)(k) + a_{k+1}(k+1) - 4a_k = 0$

$$a_{k+1} = \frac{4a_k}{(k+1)(k)+(k+1)} = \frac{4a_k}{(k+1)^2}$$

$k=0: \quad a_1 = \frac{4a_0}{1} = 4a_0$

$k=1: \quad a_2 = \frac{4a_1}{4} = a_1 = 4a_0$

$k=2: \quad a_3 = \frac{4a_2}{9} = \frac{4}{9}(4a_0) = \frac{16}{9}a_0$

$$y_1 = a_0\left(1 + 4x + 4x^2 + \frac{16}{9}x^3 + \cdots\right)$$

$$y_2 = y_1 \ln x + \sum_{n=1} b_n x^{n+r}$$

$$= y_1 \ln x + \sum_{n=1} b_n x^n$$

$$y_2' = y_1' \ln x + y_1 \frac{1}{x} + \sum_{n=1} b_n n x^{n-1}$$

$$y_2'' = y_1'' \ln x + 2y_1' \frac{1}{x} - y_1 \frac{1}{x^2} + \sum_{n=2} b_n n(n-1) x^{n-2}$$

$$x y_2'' = y_1'' x \ln x + 2y_1' - y_1 \frac{1}{x} + \sum_{n=2} b_n n(n-1) x^{n-1}$$

SUBSTITUTING INTO 1

$$y_1'' \cancel{x \ln x} + 2y_1' - \cancel{y_1 \frac{1}{x}} + \sum_{n=2} b_n n(n-1) x^{n-1}$$

$$+\cancel{y_1' \ln x} + \cancel{y_1 \frac{1}{x}} + \sum_{n=1} b_n n x^{n-1}$$

$$-4\cancel{y_1 \ln x} - \sum_{n=1} 4 b_n x^n = 0$$

SO $$2y_1' + \sum_{n=2} b_n n(n-1) x^{n-1} + \sum_{n=1} b_n n x^{n-1} - \sum_{n=1} 4 b_n x^n = 0$$

CHOOSE $a_0 = 1$ IN y_1 (ANY CHOICE WORKS)

SO EQUATION ABOVE BECOMES

$$2\left(4 + 8x + \frac{16}{3}x^2 + \cdots\right)$$

$$+ b_2(2)(1)x^1 + b_3(3)(2)x^2 + b_4(4)(3)x^3 + \cdots$$

$$+ b_1(1)x^0 + b_2(2)x^1 + b_3(3)x^2 + b_4(4)x^3 + \cdots$$

$$-4\left(b_1 x^1 + b_2 x^2 + b_3 x^3 + \cdots\right) = 0$$

x^0 TERMS: $8 + b_1 = 0 \rightarrow b_1 = -8$

x^1 TERMS: $16 + 2b_2 + 2b_2 - 4b_1 = 0$

$$b_2 = b_1 - 4 = -12$$

x^2 TERMS: $\frac{32}{3} + 6b_3 + 3b_3 - 4b_2 = 0$

$$b_3 = \frac{1}{9}\left(4b_2 - \frac{32}{3}\right) = \frac{1}{9}\left(-\frac{144}{3} - \frac{32}{3}\right) = -\frac{176}{27}$$

$$y_2 = \ln x\left(1 + 4x + 4x^2 + \frac{16}{9}x^3 + \cdots\right)$$

$$- 8x - 12x^2 - \frac{176}{27}x^3 + \cdots$$

GENERAL SOLUTION: $y = C_1 y_1 + C_2 y_2$