Definition 1.1.1 A *differential equation* (or DE for short) is an equation containing some or all of the following: an unknown function, the independent variable of the unknown function, at least one derivative of the function, and one or more constants. The constants are also called *parameters*, and the function itself is called the *dependent variable*.

Definition 1.1.2 The order of a DE is most number of derivatives of the dependent variable taken in the DE.

Example 1.1.1 The following examples help illustrate these definitions:

<table>
<thead>
<tr>
<th>Differential Equation</th>
<th>Dependent Variable</th>
<th>Independent Variable</th>
<th>Parameter(s)</th>
<th>Order</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. $x(t)'' = -\frac{c}{m}x(t)' - g$</td>
<td>$x$</td>
<td>$t$</td>
<td>$c, m, g$</td>
<td>2</td>
</tr>
<tr>
<td>2. $y'' + 4y = \sin(t)$</td>
<td>$y$</td>
<td>$t$</td>
<td>none</td>
<td>2</td>
</tr>
<tr>
<td>3. $y' + ay = e^{-x}$</td>
<td>$y$</td>
<td>$x$</td>
<td>$a$</td>
<td>1</td>
</tr>
<tr>
<td>4. $x' = ax(1 - x/b)$</td>
<td>$x$</td>
<td>$t$</td>
<td>$a, b$</td>
<td>1</td>
</tr>
</tbody>
</table>

Definition 1.1.3 A *solution* to a DE is a function that when substituted into the DE for the dependent variable, results in a true statement (meaning both sides of the equation are the same) for all values of the independent variable.

For example to show that the function $y = e^{-3x}$ is a solution to the DE $y' = -3y$ we

1. Take the derivative of the function to get $y' = -3e^{-3x}$
2. Substitute both $y$ and $y'$ into the DE and get $-3e^{-3x}$ on both sides, so the function is a solution to the DE.

Definition 1.1.4 A *general solution* of a first order DE is a solution with one constant that does not appear in the original differential equation. This is also called a *one-parameter family of solutions*. When a particular value of the constant is chosen, we get a *particular solution*.

For example the function $y = Ce^x$ is the general solution to the DE $y' = y$. Each value of $C$ gives a different solution. The function $y = 5e^x$ is a particular solution to the DE $y' = y$ corresponding to $C = 5$. 
Definition 1.1.5 An initial value problem (or IVP for short) consists of a differential equation or system of differential equations, along with a sufficient number of initial conditions to determine the arbitrary constants in the general solution to the differential equation. When there is more than one initial condition given, they must all be given at the same value of the independent variable.

A solution to an initial value problem is a function which satisfies both the differential equation(s) and the initial condition(s).

For example \( y' = 2y, y(0) = 4 \) is an IVP. The function \( y = Ce^{2x} \) is a general solution to the DE \( y' = 2y \). We can use the initial condition \( y(0) = 4 \) to find the constant \( C \) as follows: substitute \( x = 0 \) and \( y = 4 \) into the general solution to get \( 4 = Ce^{2(0)} \) which becomes \( C = 4 \). Now the function \( y = 4e^{2x} \) solves both the DE \( y' = 2y \) and the initial condition \( y(0) = 4 \), therefore it is a solution to the IVP.

The TI89 desolve command gives the general solution to many DE's (those that have an explicit solution). The same command can be used to solve many initial value problems. The syntax for a general solution (given DE only) is

\[
\text{desolve}(\text{DE}, \text{independent var}, \text{dependent var})
\]

A numerical approach: Euler’s method

Euler’s Method for First-Order IVP’s

Given a differential equation \( y' = f(x, y) \) and an initial condition \( y(x_0) = y_0 \), calculate the points \((x_1, y_1), \ldots, (x_n, y_n)\) using

\[
x_{i+1} = x_i + \Delta x \\
y_{i+1} = y_i + f(x_i, y_i) \Delta x
\]

for \( i = 0, \ldots, n - 1 \) where \( \Delta x \) is a small positive number called the step size.

<table>
<thead>
<tr>
<th>( i )</th>
<th>( x_i )</th>
<th>( y_i )</th>
<th>( f(x_i, y_i) = y_i - x_i )</th>
<th>( y_{i+1} = y_i + f(x_i, y_i) \Delta x )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0.5</td>
<td>( 0.5 - 0 = 0.5 )</td>
<td>( 0.5 + 0.5(0.5) = 0.75 )</td>
</tr>
<tr>
<td>1</td>
<td>0.5</td>
<td>0.75</td>
<td>( 0.75 - 0.5 = 0.25 )</td>
<td>( 0.75 + 0.25(0.5) = 0.875 )</td>
</tr>
<tr>
<td>2</td>
<td>1.0</td>
<td>0.875</td>
<td>( 0.875 - 1 = -0.125 )</td>
<td>( 0.875 - 0.125(0.5) = 0.8125 )</td>
</tr>
<tr>
<td>3</td>
<td>1.5</td>
<td>0.8125</td>
<td>( 0.8125 - 1.5 = -0.6875 )</td>
<td>( 0.8125 - 0.6875(0.5) = 0.46875 )</td>
</tr>
<tr>
<td>4</td>
<td>2.0</td>
<td>0.46875</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

| Table 1.1: Euler’s method for \( y' = y - x \) |

This table shows that for the DE \( y' = y - x \) and initial condition \( y(0) = 0.5 \), we can estimate \( y(2) \) to be 0.46875. Smaller step sizes give more accurate estimates.
A geometrical approach: Slope Fields

1. Choose a rectangular array of points in some region of the \(xy\)-plane.
2. At each point in the array, \((\tilde{x}, \tilde{y})\), calculate the slope of the solution curve, \(f(\tilde{x}, \tilde{y})\).
3. At each point, sketch a small straight line with the associated slope. These lines are called slope marks. The resulting graph is called a slope field.
4. Sketch approximate solution curves by drawing curves that are approximately tangent to each slope mark the curve passes near.

\[
\begin{array}{cccc}
x & 2 & -1 & 0 & 1 & 2 \\
2 & 4 & 3 & 2 & 1 & 0 \\
1 & 3 & 2 & 1 & 0 & -1 \\
0 & 2 & 1 & 0 & -1 & -2 \\
-1 & 1 & 0 & -1 & -2 & -3 \\
-2 & 0 & -1 & -2 & -3 & -4 \\
\end{array}
\]

**Figure 1.6:** Table of slopes and slope field for \(y' = y - x\)

Solution curves follow the slope marks (both forward and backward).

**Figure 1.8:** Slope field for \(y' = 3y - 100t\)

*(Newton’s law of cooling)*

\(T\) is the temperature of an object in a room of ambient temperature \(A\), and \(t\) is time.
\( \frac{dT}{dt} \) is directly proportional to \((A - T)\). This yields the DE

\[
\frac{dT}{dt} = k(A - T)
\]

where \(k > 0\) is a constant. This DE is known as Newton's law of cooling. It can be used to describe the temperature of a hot object cooling, or a cool object warming.

The general solution to the DE is \( T = A + Ce^{-kt} \)

Population models

The basic population model

A modeling assumption that is often used for population growth is that the rate of growth of a population is proportional to the size of the population (see Example 1.1.7). Notice that we are using the simplest possible proportionality model. Letting \(y(t)\) be the size of the population at time \(t\), we get the differential equation

\[
y' = ky
\]

(Basic population model)

The general solution is \( y = Ce^{kt} \)

The logistic population model

This model takes into account a carrying capacity \(N\). If we let 'a' represent the small population growth rate then the DE is

\[
y' = ay(1 - \frac{y}{N})
\]

(Logistic population model)

The general solution is

\[
y = \frac{Ne^{at}}{e^{at} + C} = \frac{N}{1 + Ce^{-at}}
\]