Differential Equations

by
R. Decker and B. Albright
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Preface

The classical approach to introductory differential equations textbooks is to present techniques for analytically solving different categories of differential equations and then analyzing the solutions algebraically. In this book we take a more modern approach, utilizing software to graphically and numerically solve differential equations. The focus of this text is on the setting up, or modeling, of the equations and the analysis of their solutions.

This text is intended for a one semester introduction course to differential equations for math, science, and engineering majors. The prerequisite is two semesters of calculus. This book is intended to be used with available software such as Maple, Mathematica, Matlab, Maxima, Wolfram Alpha, TI CAS enabled calculators, and websites. Interactive java graphing applets for first-order differential equations and first-order systems of two equations are available at uhaweb.hartford.edu/rdecker/DeckerDEbook/DeckerDEbook.html (no www at the beginning). Other applets specifically related to examples in the text are located there also.
Chapter 1

Introduction to Differential Equations

In this chapter we introduce the main concepts behind differential equations, why they are important, how they can be derived (created), and how information can be extracted from them in the form of various types of solutions (exact, graphical and numerical). The rest of the text will develop these ideas further by categorizing differential equations and introducing techniques specific to those categories.

1.1 A Brief Overview of Differential Equations

The physical laws of the universe are written in the language of differential equations. The classical mechanics of Newton, Lagrange and Hamilton, the fluid mechanics of Bernoulli and Euler, and Maxwell’s theory of electricity and magnetism are all expressed via differential equations - and form much of the theoretical basis of the engineering disciplines. In the area of modern physics, Einstein’s theory of general relativity and the quantum mechanics of Schrodinger and Dirac are based on differential equations. Differential equations have invaded many other branches of science, including (but not limited to) chemistry, biology, economics and finance, and meteorology. It is no exaggeration to claim that the modern world as we know it could not have come into being without the development of this branch of mathematics.
Laplace’s dream

The beginning student may be surprised to find that differential equations can be used to predict the future - and they have a much better track record than any psychic. To quote the great mathematician Pierre-Simon Laplace

We may regard the present state of the universe as the effect of its past and the cause of its future. An intellect which at a certain moment would know all forces that set nature in motion, and all positions of all items of which nature is composed, if this intellect were also vast enough to submit these data to analysis, it would embrace in a single formula the movements of the greatest bodies of the universe and those of the tiniest atom; for such an intellect nothing would be uncertain and the future just like the past would be present before its eyes.

—Pierre Simon Laplace, A Philosophical Essay on Probabilities

The engineer and the crumpled paper

The process described by Laplace goes something like this. Imagine that an engineer has accidently knocked a crumpled piece of paper out of her third story window, 8 meters above the ground. Being a good environmentalist, she wonders how much time she has before the paper hits the ground. She immediately recalls Newton’s second law \( F = ma \). The law says that the sum of all of the forces acting on a body are equal to the mass of the body multiplied by its acceleration. There are two forces acting on the paper; the force of gravity (pulling it downward) and the force of air resistance (which acts in the opposite direction of the motion).

Working furiously, she assigns the variable \( x \) to the position of the paper (distance above the ground in meters), lets \( t \) (in seconds) represent the time elapsed since she dropped the paper, and recalls from calculus that acceleration is the second derivative of position. Newton’s second law becomes \( F = mx'' \). She also knows that the force of gravity is given by \( mg \) where \( m \) is the mass of the body and \( g \) is the acceleration due to gravity (in meters per second\(^2 \)).

---

\(^1\) Laplace, Pierre Simon, A Philosophical Essay on Probabilities, translated into English from the original French 6th ed. by Truscott, F.W. and Emory, F.L., Dover Publications (New York, 1951) p.4

\(^2\) Due to the development of quantum theory in the 1920’s and 1930’s, this statement must be modified a bit - the differential equations of quantum mechanics make predictions about the probabilities of certain events occurring, at least on a microscopic scale. The spirit of the statement still holds, as extremely accurate predictions of such probabilities can be made.
She also assumes that the force due to air resistance is proportional to the velocity (this is a common assumption). This means that the force due to air resistance is equal to $cx'$ where $c > 0$ is a constant. If she takes the upward direction to be positive, Newton’s second law yields the equation

$$-mg - cx' = mx''$$

(the negative sign on the $cx'$ term reflects the idea that the force of friction is opposite to the direction of motion, so that if the paper is falling, $x'$ is negative and hence $-cx'$ points in the upward direction).

Rearranging this equation and dividing by $m$ our engineer obtains the equation

$$x'' = -\frac{c}{m}x' - g. \quad (1.1)$$

This equation, called a \textit{differential equation}, describes a relationship between the paper’s velocity $x'(t)$ as a function of time, its acceleration $x''(t)$ and the constants $c$, $m$, and $g$.

The engineer also knows two other pieces of information. The fact that the window is 8 meters above the ground means that $x(0) = 8$ (taking $t = 0$ to be the time the paper starts its fall). Also, the downward velocity of the paper is initially zero, since the paper is knocked off a stationary surface. Thus $x'(0) = 0$. The equations $x(0) = 8$ and $x'(0) = 0$ are called \textit{initial conditions}, and a DE along with one or more initial conditions is called an \textit{initial value problem}.

The engineer needs to estimate the values of the constants $c$, $m$, and $g$ in order to get a good prediction of when the paper hits the ground. The last is easy, as it is well know that the acceleration of gravity is $9.8 \text{ meters/second}^2$. Also, she knows the mass of one piece of paper is about 4.5 grams, or 0.0045 kilograms. The value of $c$, the proportionality constant for air resistance is harder, but fortunately she is an airplane designer, and has measured this constant for many different objects, including crumpled paper. The value is about $c = 0.01 \text{ newtons/meter/second}$. Differential equation (1.1) with the values of the constants substituted in becomes

$$x'' = -2x' - 9.8. \quad (1.2)$$

Now comes the key step. The engineer wants to find a function that solves the initial value problem. Specifically, she wants a function $x(t)$ that solves the differential equation (this means the second derivative of $x$ must equal $-2$ times the first derivative minus 9.8) and satisfies the conditions $x(0) = 8$ and $x'(0) = 0$ (how this is done in general is the subject of much of the rest of this text). This function is called a \textit{solution} to the initial value problem.
Using her knowledge of differential equations, she obtains the following solution:

\[ x(t) = -2.45 \exp(-2t) - 4.9t + 10.45 \]

This function predicts the height above the ground of the crumpled paper for any value of \( t \) (in seconds). One can easily verify that this function solves the differential equation and satisfies the initial conditions (this will be done later). This function predicts that after 1 second, for instance, the height of the paper is

\[ x(1) = -2.45 \exp(-2(1)) - 4.9(1) + 10.45 \approx 5.22 \text{ meters}. \]

The original question is, “When does the paper hit the ground?” At the time the paper hits the ground, the height is 0 meters. So to determine the time the paper hits the ground she needs to solve the equation

\[ -2.45 \exp(-2t) - 4.9t + 10.45 = 0, \]

which cannot be solved algebraically. A numerical method of solution, such as Newton’s method from calculus, is required. Such numerical solutions are built into Computer Algebra System (CAS) software. Using available software she quickly obtains the solution \( t = 2.126 \) seconds, accurate to three decimal places.

The entire process described above has taken the engineer only about a second (if you doubt this is possible, just watch any episode of the television show “Numb3rs”). Thus she still has time to save the ground from litter. She leans out the window and shouts to a bystander below, “Could you please catch that piece of paper for me?” At precisely 2.1 seconds after the paper began its fall, and with just 0.026 seconds to spare, the bystander reaches out and grabs the crumpled paper, thus saving the world from one more piece of litter. See Figure 1.1 for a graphical visualization of this solution.

**Terminology**

We begin with a definition of a *differential equation*.

**Definition 1.1.1** A *differential equation* (or DE for short) is an equation containing some or all of the following: an unknown function, the independent variable of
the unknown function, at least one derivative of the function, and one or more constants. The constants are also called parameters, and the function itself is called the dependent variable.

Every differential equation contains at least one derivative of the unknown function. These derivatives could be the first derivative, the second derivative, etc., or any combination of these. This leads to the definition of the order of a differential equation.

**Definition 1.1.2** The order of a differential equation is the number of the highest derivative that appears in the equation.

**Example 1.1.1** The following examples help illustrate these definitions:

<table>
<thead>
<tr>
<th>Differential Equation</th>
<th>Dependent Variable</th>
<th>Independent Variable</th>
<th>Parameter(s)</th>
<th>Order</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. $x(t)'' = -\frac{c}{m}x(t)' - g$</td>
<td>$x$</td>
<td>$t$</td>
<td>$c, m, g$</td>
<td>2</td>
</tr>
<tr>
<td>2. $y'' + 4y = \sin(t)$</td>
<td>$y$</td>
<td>$t$</td>
<td>none</td>
<td>2</td>
</tr>
<tr>
<td>3. $y' + ay = e^{-x}$</td>
<td>$y$</td>
<td>$x$</td>
<td>$a$</td>
<td>1</td>
</tr>
<tr>
<td>4. $x' = ax(1 - x/b)$</td>
<td>$x$</td>
<td>$t$</td>
<td>$a, b$</td>
<td>1</td>
</tr>
</tbody>
</table>
Notice that in equation 1, the independent variable is given along with the dependent variable as \( x(t) \). In the other three equations, this is not the case. In equations 2 and 3, we must use the context to determine which variable is which. In equation 4, no independent variable is explicitly given. By convention, when the dependent variable is \( x \), we typically use \( t \) as the independent variable. When the dependent variable is \( y \) and no independent variable is explicitly given, we typically use \( x \) as the dependent variable.

**Solutions to Differential Equations**

In the crumpled paper scenario we used the term *solution* to describe the function \( x(t) = -2.45 \exp(-2t) - 4.9t + 10.45 \). Now we define what we mean by a solution to a DE.

**Definition 1.1.3** A *solution* to a DE is a function that when substituted into the DE for the dependent variable, results in a true statement (meaning both sides of the equation are the same) for all values of the independent variable.

It should be stressed that the substitution mentioned in this definition involves substituting both the function and its derivative(s) into the DE. Verifying that a claimed solution really is a solution to a DE simply involves calculating the appropriate derivatives, substituting, and simplifying, as illustrated in the next example.

**Example 1.1.2** The crumpled paper scenario involved solving the DE

\[
x'' = -2x' - 9.8
\]

We claimed that

\[
x(t) = -2.45 \exp(-2t) - 4.9t + 10.45
\]

solves this differential equation. Verify that this is indeed a solution.

**Solution:** This DE involves both the first and second derivatives of \( x \). First we calculate these:

\[
x' = 4.9 \exp(-2t) - 4.9
\]
\[
x'' = -9.8 \exp(-2t).
\]
Then we substitute these derivatives into the DE and verify that both sides really are equal:

\[
-9.8 \exp(-2t) = -2(4.9 \exp(-2t) - 4.9) - 9.8
\]

\[
-9.8 \exp(-2t) = -9.8 \exp(-2t) + 9.8 - 9.8
\]

\[
-9.8 \exp(-2t) = -9.8 \exp(-2t).
\]

Note that we put question marks over the equal signs because we are not sure that the two sides really are equal, until the last equation. Since this last equation is indeed true, we conclude that \(x(t) = -2.45 \exp(-2t) - 4.9t + 10.45\) is a solution.

Verifying that a given function is a solution is relatively easy. Finding a solution is another issue. Sometimes, as illustrated in the next two examples, we can find a solution by reasoning with our knowledge of calculus and making educated guesses.

**Example 1.1.3** Find a solution to the DE \(y' = y\).

**Solution.** Note that the differential equation expressed in words says “the derivative of a function is equal to function you started with.” We should ask ourselves if we know of function with this property. From calculus we know that the only function that is its own derivative is the exponential function \(y = e^x\).

We now check our guess. The derivative of \(y = e^x\) is \(y' = e^x\). Substituting \(e^x\) in for both \(y\) and \(y'\) in the differential equation \(y' = y\) we get

\[
e^x = e^x,
\]

which is clearly a true statement. This verifies our solution.

In the previous example, it may have been a surprise to see an exponential function appear as the solution to a differential equation that itself had no exponential function in it. It is often the case that a solution to differential equations looks nothing like the DE from which it came.

**Example 1.1.4** Find a solution to the DE \(y'' = -y\).

**Solution.** In words, this DE says, “the second derivative of a function is equal to the negative of the function.” We might consider the exponential function \(y = e^x\), but this does not work this time, as its second derivative is \(e^x\), not \(-e^x\). To get a negative sign involved, we might try \(y = e^{-x}\). But then \(y' = -e^{-x}\) and \(y'' = e^{-x}\) which is the original function and not the negative of it.
So let’s consider something completely different. Consider the trigonometric function $y = \cos(x)$. Its first and second derivatives are

$$y' = -\sin(x) \quad \text{and} \quad y'' = -\cos(x).$$

Substituting these derivatives into the DE $y'' = -y$ we get $-\cos(x) = -\cos(x)$, which is a true statement. This verifies the solution. Similar calculations verify that $y = \sin(x)$ also is a solution.

Comparison of algebraic equations and differential equations

A differential equation is an equation involving an unknown function. In the equations we solved in high school algebra, the unknown was a number. Such equations are called algebraic equations. For example, the equation

$$2x + 1 = 7$$

is an algebraic equation. To solve this equation, we “do the opposite of what is being done to the unknown” by subtracting 1 from both sides, and then dividing by 2 to get $x = 3$. To check this solution, we substitute 3 in for $x$ in the equation $2x + 1 = 7$ to get $2(3) + 1 = 7$, which is a true statement.

A solution to an algebraic equation is a number, whereas the solution to a differential equation is a function. To check a solution both cases we substitute the claimed solution into the equation, and if we get a true statement then we have shown that the solution is correct. With simple algebraic equations we may be able to guess the solution, but for more complicated ones we need algebra. Differential equations are similar. For very simple DE’s, as the ones we encountered in the previous two examples, we were able to guess a solution. But more generally we will need some techniques (using calculus in addition to algebra) for coming up with solutions. Throughout this text we will develop techniques for doing just this.

Solving a DE problem can involve solving both DE’s and algebraic equations. Consider the crumpled paper scenario. To find when the paper hit the ground, we first had to solve the DE

$$x'' = -2x' - 9.8,$$

and then we had to solve the algebraic equation

$$-2.45 \exp(-2t) - 4.9t + 10.45 = 0.$$
In some cases, a DE can be converted into an algebraic equation which can then be solved using algebraic techniques (see Chapter 5). In this text we assume that the reader can obtain solutions to algebraic equations when needed, even when (as in the crumpled paper example) the solution cannot be obtained using standard algebraic techniques, using available software as needed.

**General solutions to first-order differential equations**

When performing antiderivatives in calculus, we always have an arbitrary constant $+C$ at the end. For example, the general power rule is

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1} + C.$$  

Students often think of the $+C$ as a meaningless minor technicality. When solving DE’s, the $+C$ is extremely important and cannot be ignored.

For example, consider the first order DE

$$y' = 2x.$$  

The unknown in this equation is the function $y(x)$. To solve for $y$, we can use the general principle for solving algebraic equations: Do the opposite of what is being done to the unknown. In this case, the derivative is being done to the unknown. The opposite of the derivative is the antiderivative, so we take the antiderivative of both sides of the equation:

$$\int y' \, dx = \int 2x \, dx \implies y = x^2 + C.$$  

Notice that we integrate with respect to $x$ because this is the independent variable. This gives the explicit solution $y = x^2 + C$. Because this solution involves an arbitrary constant, this solution is called a general solution.

We can choose any value of the arbitrary constant and have a solution. For instance,

$$y = x^2 + 1, \quad y = x^2 - 2.6, \quad \text{and} \quad y = x^2,$$

corresponding to $C = 1$, $-2.6$, and $0$, respectively, are all solutions to the DE $y' = 2x$ (the reader should verify this). These solutions are called particular solutions.
Definition 1.1.4 A *general solution* of a first order DE is a solution with one constant that does not appear in the original differential equation. This is also called a *one-parameter family of solutions*. When a particular value of the constant is chosen, we get a *particular solution*.

In some texts, the term *general solution* is reserved only for those situations where every possible solution to the differential equation corresponds to some value of the arbitrary constant. In this text, we use the term general solution in a slightly different way. General solutions to higher-order equations and systems of equations will be discussed in later chapters.

Not all general solutions involve a constant added to the end, as illustrated in the next example.

Example 1.1.5 Show that \( y = 2e^x \), \( y = -3e^x \), and \( y = 0 \) are all solutions to the DE \( y' = y \) and find a general solution to this DE.

Solution. The derivatives of these claimed solutions are \( 2e^x \), \(-3e^x \), and \( 0 \), respectively. Notice that in each case, the derivative equals the function. Thus these are indeed solutions.

To find a general solution, note that all these solutions are of the form \( Ce^x \) where \( C \) is a constant (in the solution \( y = 0 \), the constant is \( C = 0 \)). Therefore, a general solution is \( y = Ce^x \).

We can picture a general solution to a first-order differential equation by graphing the solution for several values of the constant. Such graphs are called *solution curves*. Figure 1.2 shows the solution curves to \( y' = y \) corresponding to \( C = -2, -1, 0, 1, \) and \( 2 \).

The next example illustrates that a general solution does not always describe all of the solutions to a DE.

Example 1.1.6 Show that a general solution to the differential equation \( P' = P^2 \) is

\[
P = \frac{-1}{C + t}.
\]

Also show that \( P = 0 \) is also a solution that does not correspond to any real value of \( C \).
Solution. Using the quotient rule, the derivative of the claimed solution is

\[ P' = \frac{(C + t) \cdot 0 - (-1) \cdot 1}{(C + t)^2} = \frac{1}{(C + t)^2}. \]

Substituting \( P \) and its derivative into the DE \( P' = P^2 \) yields

\[ \frac{1}{(C + t)^2} = \left( \frac{-1}{C + t} \right)^2, \]

which is a true statement. This verifies the solution. Note that \( P = 0 \) is also a solution because the derivative of this solution is \( P' = 0 \). Thus \( P' = P \) is a true statement. However, no value of \( C \) will make

\[ \frac{-1}{C + t} = 0 \quad \text{for all } t. \]

(One might argue that \( C = \infty \) works, but \( \infty \) is not a real number.) Thus the solution \( P = 0 \) is not “part of” the general solution \( P = -1/(C + t) \).
Initial value problems

The inclusion of an arbitrary constant in a general solution means a DE has infinitely many solutions. Laplace’s dream includes the desire to predict the future with certainty. An infinite number of solutions means an infinite number of predictions. So how can we fulfill Laplace’s dream?

Laplace’s quote contains an additional idea that we have overlooked to this point. In part he says,

An intellect which at a certain moment would know all forces that set nature in motion, and all positions of all items of which nature is composed ... for such an intellect nothing would be uncertain and the future just like the past would be present before its eyes.

It’s the “all positions” part that we have not taken into account yet. Specifically, we have not taken into account the starting positions. In terms of the DE, we have not taken into account the value of the unknown function at the starting value of the independent variable. This condition is called an initial condition. So to predict the future, we need to know both the differential equation and the initial conditions.

For a first-order differential equation, an initial condition consists of the value of the dependent variable given at some value of the independent variable. A DE along with an initial condition is called an initial value problem. The initial condition is used to find the value of the arbitrary constant, yielding a particular solution.

Example 1.1.7 We have shown that a general solution of the DE $y' = y$ is $y = Ce^x$. Suppose we are told that $y = 3$ when $x = 0$, that is $y(0) = 3$. Use this initial condition to find a particular solution.

Solution. We substitute $y = 3$ and $x = 0$ into the general solution to get

$$3 = Ce^0 = C,$$

which determines the value of $C$. Thus the particular solution is $y = 3e^x$.

Initial value problems occur frequently in applications, as illustrated in the next example.

Example 1.1.8 (Population Dynamics) An important area of science that crosses several academic boundaries is the study of population growth. Mathematicians, biologists, ecologists and others work together to create mathematical models that describe the growth
of populations (animal, human, plant, microbial, etc.). Though such models may lack the predictive precision of Newton’s second law, they can be very useful to scientists trying to manage natural resources.

The simplest population model is based on the assumption that the rate of growth of the population is proportional to the size of the population. This assumption is simply saying that large populations grow at a faster rate than small populations. If \( y(t) \) represents the number of organisms at time \( t \), then this assumption yields the differential equation

\[
y' = ky
\]

where \( k \) is a constant called the \textit{growth constant} or \textit{growth rate}.

Suppose a population of bacteria in a Petri dish grows according to this DE with growth constant \( k = 3 \), where time \( t \) is in days and \( y(t) \) is measured in hundreds of bacteria. Suppose that there were initially 1,000 organisms in the dish. Show that \( y = Ce^{3t} \) is a general solution of this DE, and use the initial condition to predict how many bacteria there will be in 2 days.

\textbf{Solution.} To verify the general solution, note that its derivative is

\[
y' = 3Ce^{3t}.
\]

Substituting the claimed solution and its derivative into the DE \( y' = 3y \), we get \( 3Ce^{3t} \) on both sides, verifying that \( y = Ce^{3t} \) is a general solution.

If we take the initial time to be \( t = 0 \), then the initial condition is \( y(0) = 10 \). To find \( C \), we substitute \( t = 0 \) and \( y = 10 \) into \( y = Ce^{3t} \) to get

\[
10 = Ce^{3(0)} = C,
\]

which yields the particular solution \( y = 10e^{3t} \). This solution can be used to make predictions. After \( t = 2 \) days we predict that we will have

\[
y = 10e^{3(2)} \approx 4034.3 \text{ thousand bacteria}.
\]

We must stress that such a prediction is at best an approximation, based on the assumption that the population is described by the DE \( y' = 3y \). If this assumption is not accurate, then our prediction is not accurate, regardless of the rigor of our mathematics.
Higher order differential equations

Our discussion of general solutions, arbitrary constants, and initial conditions has so far been restricted to first-order differential equations. To summarize, for a first-order differential equation, the general solution will contain one arbitrary constant, and we therefore need one initial condition in order to determine that constant.

In our discussion of general solutions of first-order DE’s, we saw that the arbitrary constant comes from integrating to “undo” the derivative. A second-order DE involves two derivatives. So, informally, we will have to integrate twice to solve it. This results in two arbitrary constants in a general solution. Finding the values of these constants requires the use of two initial conditions.

Generically, a general solution to an $n^{th}$ order DE will contain $n$ arbitrary constants and will require $n$ initial conditions to find a particular solution. These initial conditions can be conditions on the value of the unknown function or on the value(s) of its derivative(s). The next example illustrates this idea.

**Example 1.1.9** In the crumpled paper scenario, we found a particular solution to the second-order DE $x'' = -2x' - 9.8$. Show that $x(t) = -0.5C_1e^{-2t} - 4.9t + C_2$ is a general solution to this DE. Then use the initial conditions $x(0) = 8$ and $x'(0) = 0$ to determine $C_1$ and $C_2$. Compare this result to the particular solution found by the engineer.

**Solution.** We take two derivatives of $x(t) = -0.5C_1e^{-2t} - 4.9t + C_2$ to get

$$x'(t) = C_1e^{-2t} - 4 \quad \text{and} \quad x''(t) = -2C_1e^{-2t}.$$  

Substituting these into the DE $x'' = -2x' - 9.8$

$$-2C_1e^{-2t} = -2(C_1e^{-2t} - 4.9) - 9.8,$$

which is a true statement. This verifies the general solution.

Next we use the initial condition $x'(0) = 0$ to find $C_1$ by substituting $t = 0$ and $x' = 0$ into $x'(t) = C_1e^{-2t} - 4.9$:

$$0 = C_1 - 4.9 \quad \Rightarrow \quad C_1 = 4.9.$$  

Finally, we use the initial condition $x(0) = 8$ to find $C_2$ by substituting $t = 0$, $x = 8$, and $C_1 = 4.9$ into $x(t) = -0.5C_1e^{-2t} - 4.9t + C_2$:

$$8 = -0.5(4.9) + C_2 \quad \Rightarrow \quad C_2 = 10.45.$$
Thus the function that satisfies the differential equation and both initial conditions is

\[ x(t) = -0.5(4.9)e^{-2t} - 4.9t + 10.45 = -2.45e^{-2t} - 4.9t + 10.45, \]

which is the same solution found by the engineer.

### Systems of differential equations

Many real-world scenarios involve two or more unknown functions that “interact.” Describing these scenarios requires two or more DE’s. Such descriptions are called systems of differential equations. A solution to a system of n DE’s consists of n functions which when substituted into the equations results in true statements.

Systems of first-order DE’s are important, in part, because an \( n^{th} \)-order DE can be rewritten as an equivalent system of n first-order DE’s. The system may be easier to solve than the original single DE.

**Example 1.1.10** The equations

\[
\begin{align*}
x' &= y, \\
y' &= -x
\end{align*}
\]

form a system of two first-order differential equations. Show that the functions

\[
\begin{align*}
x(t) &= C_1 \cos(t) + C_2 \sin(t), \\
y(t) &= -C_1 \sin(t) + C_2 \cos(t)
\end{align*}
\]

form a general solution to this system. In addition, find the particular solution satisfying the initial conditions \( x(0) = 1 \) and \( y(0) = -1 \).

**Solution.** The derivatives of these functions are

\[
\begin{align*}
x'(t) &= -C_1 \sin(t) + C_2 \cos(t), \\
y'(t) &= -C_1 \cos(t) - C_2 \sin(t) \\
&= -(C_1 \cos(t) + C_1 \sin(t)).
\end{align*}
\]

This shows that \( x' = y \) and \( y' = -x \), and hence \( x(t) \) and \( y(t) \) form a general solution.

Substituting \( t = 0, \ x = 1, \) and \( y = -1 \) into the general solution for \( x(t) \) and \( y(t) \) we get...
the two equations
\[ 1 = C_1 \cos(0) + C_2 \sin(0), \]
\[ -1 = -C_1 \sin(0) + C_2 \cos(0), \]
which simplify to \( C_1 = 1 \) and \( C_2 = -1 \). Thus
\[ x(t) = \cos(t) - \sin(t), \]
\[ y(t) = -\sin(t) - \cos(t) \]
is the particular solution.

In the previous example we were given a system of DE’s and enough initial conditions to find the values of the constants in the general solution. This leads to our definition of an initial value problem.

**Definition** 1.1.5 An initial value problem (or IVP for short) consists of a differential equation or system of differential equations, along with a sufficient number of initial conditions to determine the arbitrary constants in the general solution to the differential equation. When there is more than one initial condition given, they must all be given at the same value of the independent variable.

A solution to an initial value problem is a function which satisfies both the differential equation(s) and the initial condition(s).

**Example** 1.1.11 Show that the given function is a solution to the given IVP.

<table>
<thead>
<tr>
<th>Number</th>
<th>Function(s)</th>
<th>IVP</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>( y = 3e^{10x} )</td>
<td>( y' = 10y, \ y(0) = 3 )</td>
</tr>
<tr>
<td>2.</td>
<td>( y = 3 \cos(t) )</td>
<td>( y'' + y = 0, \ y(0) = 3, \ y'(0) = 0 )</td>
</tr>
<tr>
<td>3.</td>
<td>( x = \sin(t), \ y = \cos(t) )</td>
<td>( x' = y, \ y' = -x, \ x(0) = 0, \ x(0) = 1 )</td>
</tr>
</tbody>
</table>

**Solution.** Verifying a solution to an IVP requires verifying that the solution satisfies the DE and the initial conditions.
<table>
<thead>
<tr>
<th>Number</th>
<th>Verify DE</th>
<th>Verify IC</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>$y' = 30e^{10x} = 10 \cdot 3e^{10x} = 10y$</td>
<td>$y(0) = 3e^{10(0)} = 3$</td>
</tr>
<tr>
<td>2.</td>
<td>Note that $y' = -3\sin(t)$ and $y'' = -3\cos(t)$ so $y'' + y = -3\cos(t) + 3\cos(t) = 0$.</td>
<td>$y(0) = 3\cos(0) = 3$</td>
</tr>
<tr>
<td>3.</td>
<td>$x' = \cos(t) = y$ and $y' = -\sin(t) = -x$</td>
<td>$x(0) = 0$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$y(0) = 1$</td>
</tr>
</tbody>
</table>

### Exercises

**Directions:** For each equation 1-6 below, determine its order. Also, name the independent variable, the dependent variable, and any parameters in the equation. If it is not clear what the independent variable is, choose one (generally $t$ or $x$ if it is not being used as a dependent variable). Assume any letters that are not $t$, $x$, or $y$ (or the dependent variable) are parameters.

1.1.1 $y' = ky$
1.1.2 $dP/dt = rP(1 - P/N)$
1.1.3 $x'' + 2x' + 2x = \sin(t)$
1.1.4 $\theta'' + \theta' + k\sin\theta = 0$
1.1.5 $\frac{d^4y}{dx^4} + 4y = 0$
1.1.6 $T'(t) = k(A - T(t))$

Guess a solution to each DE below. It does not have to be a general solution. Check your guess to make sure that it works.

1.1.7 $y' = -y$
1.1.8 $y' = -5y$
1.1.9 $y'' = y$
1.1.10 $y'' = 3y$
1.1.11 $y'' = -3y$
1.1.12 $y^{(4)} = y$ (here $y^{(4)}$ means the fourth derivative of $y$)
For each of the equations, or system of equations below, show that the given function(s) form a solution.

1.1.13 Equation is \( y' = y + 1 \), solution is \( y(t) = e^t - 1 \)

1.1.14 Equation is \( y' = y + \sin(t) \), solution is \( y(t) = e^t - \frac{1}{2} \sin t - \frac{1}{2} \cos t \)

1.1.15 Equation is \( x'' + 4x' + 4x = 0 \), solution is \( x(t) = e^{-2t} \)

1.1.16 Equation is \( x'' + 4x' + 4x = 0 \), solution is \( x(t) = te^{-2t} \)

1.1.17 System is \( x' = y, \quad y' = 4x \), solution is \( x(t) = -\frac{1}{2}e^{-2t}, \quad y(t) = e^{-2t} \)

1.1.18 System is \( x' = y, \quad y' = 4x \), solution is \( x(t) = \frac{1}{2}e^{2t}, \quad y(t) = e^{2t} \)

In each problem below, a differential equation, a general solution to the differential equation and one or more initial conditions are given. First show that the given function is in fact a solution, then use the initial condition(s) to determine the (integration) constants.

1.1.19 Equation is \( y' = y + 1 \), general solution is \( y(t) = Ce^t - 1 \), initial condition is \( y(0) = 4 \).

1.1.20 Equation is \( y' = y + \sin(t) \), general solution is \( y(t) = Ce^t - \frac{1}{2} \sin t - \frac{1}{2} \cos t \), initial condition is \( y(0) = 0 \).

1.1.21 Equation is \( x'' - 4x = 0 \), general solution is \( x(t) = C_1e^{-2t} + C_2e^{2t} \), initial conditions are \( x(0) = 1, \quad x'(0) = 0 \).

1.1.22 Equation is \( x'' + 4x' + 4x = 0 \), general solution is \( x(t) = C_1e^{-2t} + C_2te^{-2t} \), initial conditions are \( x(0) = 1, \quad x'(0) = 0 \).

1.1.23 System is \( x' = y \quad y' = 4x \), general solution is \( x(t) = \frac{1}{2}C_1e^{2t} - \frac{1}{2}C_2e^{-2t}, \quad y(t) = C_1e^{2t} + C_2e^{-2t} \), initial conditions are \( x(0) = 1, \quad y(0) = -1 \).

1.1.24 System is \( x' = y \quad y' = -4x \), general solution is \( x(t) = -\frac{1}{2}C_1 \cos(2t) + \frac{1}{2}C_2 \sin(2t), \quad y(t) = C_1 \sin(2t) + C_2 \cos(2t) \), initial conditions are \( x(0) = 1, \quad y(0) = -1 \).

### 1.2 Explicit, Numerical, and Graphical Solutions

In Section 1.1 we defined a DE as an equation involving an unknown function and one or more of its derivatives and a solution to a DE to be a function which, when substituted into the DE, results in a true statement for all values of the independent variable. So far we have come up with an algebraic representation of a solution (a “formula” for the dependent variable in terms of the independent variable), which we will refer to as an explicit solution. The problem is that for many DE’s, such an explicit solution cannot be found.

Solving a DE means finding the unknown function. There are three common ways of representing a function:
1. Algebraically (a “formula” for the function)
2. Numerically (a table of numerical outputs for different inputs)
3. Graphically (a plot of the outputs vs the inputs)

In cases where an explicit solution cannot be found, one can resort to approximation methods that give either approximate values of the outputs for different inputs or an approximation of the graph without having to find an explicit solution. The results are often referred to as numerical and graphical, or geometrical solutions, respectively.

In this section we define an explicit solution and describe some basic methods for finding numerical and graphical solutions.

Explicit solutions and elementary functions

Before we define an explicit solution, we define elementary functions.

**Definition 1.2.1** To say that an algebraic expression can be written in terms of elementary functions means that the expression can be built from a finite number of exponentials, logarithms, trigonometric functions and their inverses, constants, and \( n \)th roots through composition and combinations using the four elementary operations of addition, subtraction, multiplication, and division.

The next example illustrates this definition.

**Example 1.2.1** The following algebraic expressions can be expressed in terms of elementary functions:

\[
\sin(x) - x, \quad 3 \cos(2x), \quad \frac{2\sqrt{x} + \ln(2x)}{5e^x - \arctan(x^2 + 2x + 1)}, \quad \text{and} \quad 1.
\]

The following cannot be expressed in terms of elementary functions:

\[
\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt, \quad \Gamma(x) = \int_0^\infty e^{-t}t^{x-1}dt, \quad \text{and}
\]
These last three functions are very important in mathematics, and have the names Error function, Gamma function, and Bessel function (of the first kind), and are generically referred to as special functions. It can be proven that these three functions cannot be written in terms of elementary functions, but the proofs are beyond the scope of this book.\(^3\)

Elementary functions are the types of functions we are used to working with, and they are relatively intuitive to understand. Our definition of an explicit solution involves elementary functions.

**Definition 1.2.2** An explicit solution to a differential equation or initial value problem is an algebraic representation of the solution to the DE or IVP, written in terms of elementary functions, where the dependent variable is written explicitly in terms of the independent variable.

"What is considered “elementary” and what is not is somewhat arbitrary. Things could change in the future. Mathematics is an evolving discipline, not a static one. Perhaps Bessel functions will become so common in the future that they will be a button on every scientific calculator, just like sin or cos, and that they will then be classified as elementary."
Example 1.2.2 Use available software to find general solutions to the following differential equations. Also, state whether or not each solution given qualifies as an explicit solution by our definition.

1. \( y' = y(1 - y) \)
2. \( y'' + y = \sin(t) \)
3. \( y' = y^2 - t \)

Solution. The following results are those given by Wolfram Alpha. In all parts, \( c_1 \) and \( c_2 \) denote arbitrary constants.

1. \( y(x) = \frac{e^x}{c_1 + e^x} \) is an explicit solution
2. \( y(t) = c_2 \sin(t) + c_1 \cos(t) - \frac{1}{2}t \cos(t) \) is an explicit solution
3. \[ y(t) = \frac{i t^{3/2} \left( -c_1 J_{-\frac{1}{3}} \left( \frac{2}{3} i t^{3/2} \right) + c_1 J_{\frac{2}{3}} \left( \frac{2}{3} i t^{3/2} \right) - 2 J_{-\frac{1}{3}} \left( \frac{2}{3} i t^{3/2} \right) - c_1 J_{\frac{2}{3}} \left( \frac{2}{3} i t^{3/2} \right) \right) - c_1 J_{-\frac{1}{3}} \left( \frac{2}{3} i t^{3/2} \right) + J_{\frac{2}{3}} \left( \frac{2}{3} i t^{3/2} \right) \right) }{2 t \left( c_1 J_{-\frac{1}{3}} \left( \frac{2}{3} i t^{3/2} \right) + J_{\frac{2}{3}} \left( \frac{2}{3} i t^{3/2} \right) \right) } \]
   is not an explicit solution because of the use of the function \( J \), a Bessel function (see Example 1.2.1).

Note that other software may return something that looks very different, or no result at all. For part 3, for instance, the TI89 just returns the original DE, meaning that it cannot find a solution.

The main point of Example 1.2.2 is that not all explicit solutions to DE’s can be expressed in terms of elementary functions. An explicit solution given in terms of non elementary functions, such as in part 3, may be fine from a theoretical perspective, but such a description does not provide much intuitive insight into the behavior of the solution.

A numerical approach: Euler’s method

In Example 1.1.8, we solved the initial value problem

\[ \text{DE : } y' = 3y \]
\[ \text{IC : } y(0) = 10 \]
by first finding a general explicit solution to the DE, \( y = Ce^{3x} \), and then using the initial condition to find the particular solution \( y = 10e^{3x} \) (note that we changed the independent variable from \( t \) to \( x \)). We then used this particular solution to predict the future value of \( y \).

Numerical methods allow us to predict future values of the function without ever having to find an explicit solution. Euler’s method is a relatively simple numerical algorithm for doing just this to IVP’s of the form

\[
\begin{align*}
\text{DE : } y' &= f(x, y) \\
\text{IC : } y(x_0) &= y_0.
\end{align*}
\]

The right-hand side of this DE simply means that \( y' \) is given in terms of the independent variable and/or the dependent variable. Euler’s method is based on three observations:

1. The solution to this IVP is a function which has a graph that goes through the point \((x_0, y_0)\).
2. \( y' \) is the slope of the tangent line of this graph.
3. \( f(x, y) \) gives a formula for the slope of the tangent line at any point \((x, y)\) on the graph.

In Figure 1.3, the curve \( y(x) \) is the graph of the exact solution to the IVP (we don’t know the exact solution). The point \((x_0, y_0)\) is a “starting point.” We move horizontally to the right a small distance \( \Delta x \) to get a new value of the independent variable. Call this new value \( x_1 = x_0 + \Delta x \). The goal is to estimate the corresponding \( y \)-coordinate on the exact solution curve, \( y(x_1) \).

We draw the tangent line to the solution curve at the point \((x_0, y_0)\), the slope of which is \( f(x_0, y_0) \). Let \( y_1 \) be the \( y \)-coordinate of point on this tangent line corresponding to the \( x \)-coordinate \( x_1 \). Graphically we see that \( y_1 \approx y(x_1) \).

To find the value of \( y_1 \), note that slope is rise over run, so

\[
f(x_0, y_0) = \frac{\Delta y}{\Delta x} \quad \Rightarrow \quad \Delta y = f(x_0, y_0) \Delta x
\]

and

\[
y_1 = y_0 + f(x_0, y_0) \Delta x.
\]

This gives a simple formula for the approximate value of \( y(x_1) \). Graphically, the point \((x_1, y_1)\) is an approximate point on the solution curve.
Now take the new point \((x_1, y_1)\) and use it as the starting point to find another new point \((x_2, y_2)\) where

\[
\begin{align*}
x_2 &= x_1 + \Delta x, \\
y_2 &= y_1 + f(x_1, y_1) \Delta x.
\end{align*}
\]

We can repeat this process as many times as we want to find any number of approximate points on the solution curve. We write the general algorithm as follows.

**Euler’s Method for First-Order IVP’s**

Given a differential equation \(y' = f(x, y)\) and an initial condition \(y(x_0) = y_0\), calculate the points \((x_1, y_1), \ldots, (x_n, y_n)\) using

\[
\begin{align*}
x_{i+1} &= x_i + \Delta x \\
y_{i+1} &= y_i + f(x_i, y_i) \Delta x
\end{align*}
\]

for \(i = 0, \ldots, n - 1\) where \(\Delta x\) is a small positive number called the *step size*.

In problems such as Example 1.1.8, we want to know the value of \(y(b)\) where \(b\) is some given value of the independent variable. In such cases we can use the following relationship
Table 1.1: Euler’s method for $y' = y - x$

<table>
<thead>
<tr>
<th>$i$</th>
<th>$x_i$</th>
<th>$y_i$</th>
<th>$f(x_i, y_i) = y_i - x_i$</th>
<th>$y_{i+1} = y_i + f(x_i, y_i) \Delta x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0.5</td>
<td>$0.5 - 0 = 0.5$</td>
<td>$0.5 + 0.5(0.5) = 0.75$</td>
</tr>
<tr>
<td>1</td>
<td>0.5</td>
<td>0.75</td>
<td>$0.75 - 0.5 = 0.25$</td>
<td>$0.75 + 0.25(0.5) = 0.875$</td>
</tr>
<tr>
<td>2</td>
<td>1.0</td>
<td>0.875</td>
<td>$0.875 - 1 = -0.125$</td>
<td>$0.875 - 0.125(0.5) = 0.8125$</td>
</tr>
<tr>
<td>3</td>
<td>1.5</td>
<td>0.8125</td>
<td>$0.8125 - 1.5 = -0.6875$</td>
<td>$0.8125 - 0.6875(0.5) = 0.46875$</td>
</tr>
<tr>
<td>4</td>
<td>2.0</td>
<td>0.46875</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Figure 1.3 illustrates that $y_i$ is not quite equal to $y(x_i)$ (the exact value of the solution at $x = x_i$). The error

$$|y(x_i) - y_i|$$

for $i = 0, \ldots, n - 1$ is called the \textit{local truncation error}. When trying to estimate $y(b)$ using $n$ steps, the error

$$|y(b) - y_n|$$
is called the \textit{global truncation error}. In Example 1.2.3 the Euler method estimate of $y(2)$ using $n = 4$ steps was 1.46875 and the exact value of $y(2)$ was $-0.69453$ to five digits. The global truncation error is

$$|1.46875 - (-0.69453)| = 2.1633.$$

This is a very large error. This error could be decreased by making the step size $\Delta x$ smaller. However, there is a trade-off. A smaller step size requires a larger number of steps. If we were to choose a step size of $\Delta x = 0.1$, the number of steps would be

$$0.1 = \frac{2 - 0}{n} \Rightarrow n = 20.$$

This number of calculations is impractical to do by hand, but luckily Euler’s method is easy to implement on a computer. The issue of errors in Euler’s method will be discussed more in Chapter 2.

\section*{Euler’s method with software}

Euler’s method is an option when numerically solving initial value problems on the major computer algebra systems, certain calculators, and many applets including those on the author’s website. These software calculate the points $(x_1, y_1), \ldots, (x_n, y_n)$ and provide them in the form of a table. The software also plots the points and connects them with straight lines to approximate the exact solution curve.

\textbf{Example 1.2.4} Use available software, or the applet labeled Example 1.2.4 at \texttt{uhaweb.hartford.edu/rdecker/DeckerDEbook/DeckerDEbook.html} to approximate the solution the differential equation $y’ = y - x$ with the initial condition $y(0) = 0.5$ over the interval $0 \leq x \leq 2$ using step sizes of $\Delta x = 0.5, 0.1, 0.01,$ and 0.001. Compare these approximate solutions to the exact solution $y = -0.5e^x + x + 1$. What happens to the quality of the approximations as $\Delta x$ gets smaller?

\textbf{Solution}. Partial tables of the results given by the applet referenced above are shown below. The exact value of $y(2)$ is $-0.69453$. Notice that as $\Delta x$ gets smaller, the value of $y_n$ gets closer to this exact value.
\[ \Delta x = 0.5 \quad \Delta x = 0.1 \quad \Delta x = 0.01 \quad \Delta x = 0.001 \]

<table>
<thead>
<tr>
<th></th>
<th>( x_i )</th>
<th>( y_i )</th>
<th>( x_i )</th>
<th>( y_i )</th>
<th>( x_i )</th>
<th>( y_i )</th>
<th>( x_i )</th>
<th>( y_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.0</td>
<td>0.5</td>
<td>0.0</td>
<td>0.5</td>
<td>0.0</td>
<td>0.5</td>
<td>0.0</td>
<td>0.5</td>
</tr>
<tr>
<td>0.5</td>
<td>0.75</td>
<td>0.1</td>
<td>0.55</td>
<td>0.01</td>
<td>0.505</td>
<td>0.001</td>
<td>0.5005</td>
<td></td>
</tr>
<tr>
<td>0.0</td>
<td>0.875</td>
<td>0.2</td>
<td>0.595</td>
<td>0.02</td>
<td>0.50995</td>
<td>0.002</td>
<td>0.501</td>
<td></td>
</tr>
<tr>
<td>1.5</td>
<td>0.8125</td>
<td>1.9</td>
<td>-0.15795</td>
<td>1.99</td>
<td>-0.63179</td>
<td>1.999</td>
<td>-0.68815</td>
<td></td>
</tr>
<tr>
<td>2.0</td>
<td>0.46875</td>
<td>2.0</td>
<td>-0.36375</td>
<td>2.0</td>
<td>-0.65801</td>
<td>2.0</td>
<td>-0.69084</td>
<td></td>
</tr>
</tbody>
</table>

Graphs of the approximate solution curve and the exact curve for the different values of \( \Delta x \) are shown in Figure 1.4. In each graph, the red curve is the exact solution curve and the black curve is the approximation. We see that as \( \Delta x \) gets smaller, the approximate curve gets closer to the exact curve.

**Figure 1.4:** Approximate solution curves
A geometrical approach: Slope Fields

Euler’s method is a fairly simple algorithm. But it has a drawback: it requires an initial condition and thus gives us an approximation of a particular solution to the DE. In some cases we don’t know an IC, or we would like to know the behavior of the solution for different IC’s. In otherwords, we would like a description of a general solution.

The method of slope fields allows us to visualize all possible solution curves to a differential equation. That it, it gives us a graphical representation of a general solution. This method applies to first-order DE’s of the form

\[ y' = f(x, y) \]

but no initial condition is required. The key idea is the same as for Euler’s method: the right-hand side of the DE provides a formula for the slope of the solution curve at any given point. The basic steps are as follows:

1. Choose a rectangular array of points in some region of the \(xy\)-plane.
2. At each point in the array, \((\hat{x}, \hat{y})\), calculate the slope of the solution curve, \(f(\hat{x}, \hat{y})\).
3. At each point, sketch a small straight line with the associated slope. These lines are called slope marks. The resulting graph is called a slope field.
4. Sketch approximate solution curves by drawing curves that are approximately tangent to each slope mark the curve passes near.

Figure 1.5 shows a slope field with just two slope marks for the DE \(y' = y - x\) at the points \((-1, 1)\) and \((1, 1)\).

A slope field with only two slope marks does not give much information about the behavior of the solutions. So we extend this slope field using slope marks at integer values of \(x\) and \(y\) for \(-2 \leq x \leq 2\) and \(-2 \leq y \leq 2\). The calculations and resulting slope field are shown in Figure 1.6.

Using this slope field, we can now sketch a few approximate solution curves as shown in Figure 1.7. Notice how these curves are drawn to be tangent to the slope lines.

These solution curves exhibit three distinct types of behavior. To categorize these types, we first make two general observations about initial conditions and solution curves for DE’s of the form \(y' = f(x, y)\):

1. An initial condition \(y(x_0) = x_0\) is a point on the \(xy\)-plane.
Figure 1.5: Slope field for $y' = y - x$

<table>
<thead>
<tr>
<th>$x$</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
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<td>4</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>0</td>
</tr>
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<td>3</td>
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<td>0</td>
<td>-1</td>
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<td>0</td>
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<td>-1</td>
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<tr>
<td>-2</td>
<td>0</td>
<td>-1</td>
<td>-2</td>
<td>-3</td>
<td>-4</td>
</tr>
</tbody>
</table>

Figure 1.6: Table of slopes and slope field for $y' = y - x$

2. The solution curve corresponding to an initial condition will go through that point.

Thus every point on a solution curve can be thought of as a (potential) initial condition.

Next note that one solution curve appears to be a straight line. This line appears to go through the points $(-1, 0)$ and $(0, 1)$. The equation of this line is $y = x + 1$. One can easily verify that this is an explicit particular solution to the DE. We categorize the types of solution curves based on the location of the initial condition relative to this line:

1. **IC above $y = x + 1$:** Moving to the left, the solution asymptotically approaches $y = x + 1$. Moving to the right, the solution approaches $+\infty$.  

   Thus every point on a solution curve can be thought of as a (potential) initial condition.
2. **IC on** \( y = x + 1 \): The solution is \( y = x + 1 \).

3. **IC below** \( y = x + 1 \): Moving to the left, the solution asymptotically approaches \( y = x + 1 \). Moving to the right, the solution increases to a maximum, then decreases and approaches \(-\infty\). (Notice that in the table of slopes, the slopes of 0 with positive slopes to the left and negative slopes to the right indicate a maximum.)

This example illustrates how slope fields are useful for analyzing the “long-term” behavior of solutions (meaning how the solution \( y(x) \) behaves as \( x \to \pm\infty \)). This example also illustrates that, as with Euler’s method, slope fields are most useful if we have a computer or calculator do the dirty work for us. The more slope marks we calculate, the more accurate is the picture of the set of solution curves to the DE. We used only a 5 by 5 grid. Programs typically use a 15 by 15 grid (225 marks) or larger.

**Example 1.2.5** In Example 1.1.8 we saw the basic population growth model \( y' = ky \), where \( y \) is the size of the population and \( k \) is a constant. Now let’s suppose we remove members of the population at a rate \( f(t) \) where \( t \) is the independent variable (representing time). The new model is

\[
y' = ky - f(t)
\]

since the overall rate of change of the population, \( y' \), is the difference between the rate at which organisms are added, \( ky \), and the rate at which they are removed, \( f(t) \). Notice that by writing the removal rate as \( f(t) \) we are stating that the removal rate can change with time \( t \) but cannot change with the number of organisms \( y \).

Consider the bacteria in a Petri dish from Example 1.1.8 where \( k = 3 \). Suppose that an antibiotic is continuously added to the Petri dish and kills the bacteria at a rate according
to the function \( f(t) = 100t \) where \( t \) is measured in days. What will happen to the long-term population of bacteria in the dish? Does the initial population matter?

**Solution.** Note that an initial population \( y_0 \) corresponds to the initial condition \( y(0) = y_0 \). This initial condition is a point on the positive \( y \)-axis. Thus we only need to consider solution curves that “start at” such a point. Figure 1.8 shows a slope field with three such solution curves drawn. The applet corresponding to this example at [uhaweb.hartford.edu/rdecker/DeckerDEbook/DeckerDEbook.html](http://uhaweb.hartford.edu/rdecker/DeckerDEbook/DeckerDEbook.html) can be used to create a similar graph.

![Slope field for \( y' = 3y - 100t \)](image)

**Figure 1.8:** Slope field for \( y' = 3y - 100t \)

The “middle” solution curve corresponds to an initial population of about 11.11. This curve indicates that the population increases in an almost linear fashion. For initial populations above 11.11, the population quickly increases without bound. For initial populations less than 11.11, the population eventually dies out. Thus we see that the long-term population is greatly affected by the initial population, even for a very small change in it.

**Comparison of exact, numerical and graphical approaches**

We have now seen in action the “big three” methods for getting information out of a differential equation. It is time to think about strengths and weaknesses of each approach. One of the most important goals of this book is to understand which approach (or approaches) to take to answer a given question about a given differential equation.

When assessing the strengths and weaknesses of a method, we need to take into account the relative complexity of the method. Obviously, the simpler the better. We also need to consider the necessity of initial conditions. Requiring an initial condition imposes a
Explicit, Numerical, and Graphical Solutions

<table>
<thead>
<tr>
<th>Method</th>
<th>Strengths</th>
<th>Weaknesses</th>
</tr>
</thead>
<tbody>
<tr>
<td>Explicit</td>
<td>Provides the most detailed information.</td>
<td>Finding an explicit solution can be very complicated or impossible.</td>
</tr>
<tr>
<td></td>
<td>Solution can be algebraically analyzed to determine the affect of parameters and initial conditions.</td>
<td>The algebraic analysis of the solution can be very complicated.</td>
</tr>
<tr>
<td></td>
<td>Does not require initial conditions or the values of parameters.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Provides an exact solution.</td>
<td></td>
</tr>
<tr>
<td>Numerical</td>
<td>Relatively easy to calculate.</td>
<td>Parameters and initial conditions must be known.</td>
</tr>
<tr>
<td></td>
<td>Results can be used to draw an approximate solution curve.</td>
<td>Accuracy may be low. This can be improved by decreasing the step size, but then the number of steps must be increased.</td>
</tr>
<tr>
<td></td>
<td>Can be used on virtually any differential equation.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Does not require the existence of an explicit solution.</td>
<td></td>
</tr>
<tr>
<td>Graphical</td>
<td>Good for showing long-term behavior.</td>
<td>Does not show short-term behavior with much accuracy.</td>
</tr>
<tr>
<td></td>
<td>Initial conditions need not be known.</td>
<td>Parameters must be known.</td>
</tr>
<tr>
<td></td>
<td>Does not require the existence of an explicit solution.</td>
<td></td>
</tr>
</tbody>
</table>

Table 1.2: Comparison of solution methods

restriction. Another issue is the presence of parameters. It is often the case that we want to analyze how the behavior of the solution changes as the parameter(s) change. We would like the method to easily facilitate this analysis. Table 1.2 summarizes some strengths and weaknesses of these three methods.

Exercises

Directions: Determine which of the following differential equations have general solutions in terms of elementary functions and which do not, using the computer software, calculator
or website of your choice. Give the solution when you can find one.

1.2.1 \( y' = y + \sin(x) \)

1.2.2 \( y' = x + \sin(y) \)

1.2.3 \( y' + y = e^t \)

1.2.4 \( y' + y^3 = e^t \)

Use Euler’s method for each of the following. If more than 5 steps are required, use a computer or calculator. If an exact solution is possible, find it, and calculate the error in Euler’s method.

1.2.5 Given that \( y' = 3y - 100t \) and \( y(0) = 10 \), estimate \( y(1) \) using \( \Delta t = 0.25 \).

1.2.6 Given that \( y' = 3y - 100t \) and \( y(0) = 10 \), estimate \( y(1) \) using \( \Delta t = 0.01 \).

1.2.7 Given that \( y' = 3y(1 - \frac{y}{100}) - 100t \) and \( y(0) = 10 \), estimate \( y(1) \) using \( \Delta t = 0.25 \).

1.2.8 Given that \( y' = 3y(1 - \frac{y}{100}) - 100t \) and \( y(0) = 10 \), estimate \( y(1) \) using \( \Delta t = 0.01 \).

Create slope fields as directed for each of the following DE’s. Sketch in a few solution curves based on the slope field. If more than 10 slope marks are required, use a computer or calculator. What can you say about the long term behavior of the differential equation (what happens for various initial conditions)?

1.2.9 DE is \( y' = -2y \), using integer values of the variables for \( 0 \leq x \leq 2 \) and \( -1 \leq y \leq 1 \).

1.2.10 DE is \( y' = -2x \), using integer values of the variables for \( 0 \leq x \leq 2 \) and \( -1 \leq y \leq 1 \).

1.2.11 DE is \( y' = y(1 - y) \) for \( -1 \leq x \leq 5 \) and \( -1 \leq y \leq 2 \) using a 15 by 15 grid or larger.

1.2.12 DE is \( y' = y^2 - t \) for \( -1 \leq t \leq 5 \) and \( -2 \leq y \leq 2 \) using a 15 by 15 grid or larger.

For each situation below, use whatever approach (exact, numerical, graphical) you feel is most appropriate to answer the given question. When using a numerical approach use Euler’s method with step size 0.01. Use a computer algebra system (software, calculator or website) to determine an exact solution if there is one.

1.2.13 A cup of coffee at 180°F is left in a room at 70°F. Newton’s law of cooling states that the rate of change of the temperature of an object is proportional to the difference between the object’s temperature and the surrounding (ambient) temperature. Letting \( y \) represent the temperature of the coffee and letting \( t \) represent time (in, say, minutes) we get the differential equation \( y' = k(70 - y) \) where \( k \) is the proportionality constant. The initial condition is \( y(0) = 180 \).

Assume that from experiment it has been found that for a cup of coffee, we have approximately \( k = 0.05 \). Find both the temperature after 10 minutes and the long-term temperature (that is, the temperature after a very long time, that is, \( \lim_{t \to \infty} y(t) \)).
1.2.14 For the situation in problem 13, suppose we are interested in what happens to cups of coffee that start out at various temperatures (not just 180°F)? Describe what happens to cups of coffee that start out at any temperature ranging between 0°F and 200°F. Give a qualitative description for the short term, and a more quantitative description (numerical values) for what happens in the long term.

1.2.15 Recall the falling crumpled paper from Section 1.1. We made the assumption that the air resistance force was proportional to the velocity of the object, and we ended up with the differential equation $x'' = -2x' - 9.8$. This assumption may be inaccurate for some objects. A more general model for falling objects is $x'' = -kx'|x'|^{p-1}$. We are basically assuming that the air resistance force is proportional to the velocity raised to some power $p$, but instead of just replacing $-cx'$ with $-cx'|x'|^p$ we must use the more complicated $-c(x')^p$ in order to make sure that the direction of the force is opposite to the direction of the motion of the object, and that we don’t take roots of negative numbers. Switching the dependent variable to velocity $v = x'$ we get $v' = -kv|v|^{p-1} - 9.8$.

Assume that $k = 2$ as before, but now assume that $p = 1.5$ (instead of $p = 1$). The DE is now $v' = -2v|v|^{0.5} - 9.8$. If the paper starts out at 0 velocity, estimate the velocity after 1 second. Also determine the velocity after a long period of time (called the terminal velocity).

1.2.16 Find a linear function of the form $f(t) = a + bt$ that is a solution to the differential equation $y' = 3y - 100t$. To do this substitute $f(t)$ in for $y$ in the differential equation and solve for the constants $a$ and $b$. This is one of the standard techniques of differential equations; assume a form for the solution which contains arbitrary constants, substitute the proposed solution into the DE, then solve for the constants. Key idea: equate coefficients of like terms (with a polynomial the like terms are the $t^0$ terms, the $t^1$ terms, the $t^2$ terms and so on).

1.3 Mathematical Modeling with Differential Equations: General Principles

We have used the term “model” several times in the first two sections of this text. In this section we discuss what we mean by a model, some basic principles for constructing models, and a few examples of models.

We begin with a definition.

**Definition** 1.3.1 A mathematical model is a mathematical description of a real-world problem.
Mathematical models can take many forms. A model may be an algebraic equation, a
differential equation, a system of equations, an algorithm, a simulation, or any other number
of possibilities. Differential equations are often used to construct models because it is
often easier to describe the way a quantity changes than it is to describe the quantity
itself. Because of this, differential equations is one of the most applied branches in all of
mathematics.

Mathematical modeling is all about using mathematics to describe real-world problems.
The mathematics world is very precise, rigorous, and certain. None of this is true about
the real world. Because of this, it is necessary to make assumptions about the way the
real-world works in order to construct a model. Every mathematical model is based on
some set of assumptions. The results of the model are only as valid as the assumptions on
which it is based. This point cannot be over-stated.

Many mathematical models combine “laws of nature,” which have been extensively exper-
imentally verified, and assumptions which may seem reasonable, but often do not have the
predictive accuracy of the laws of nature.

**Example 1.3.1** In Section 1.1 we derived the DE \( mx'' = -cx' - mg \) to model a falling
crumpled paper. This DE is an example of a mathematical model.

We derived this equation using two main ideas:

1. Newton’s second law \( \sum F = ma \), which is a law of nature, and
2. the assumption that the force due to air resistance is proportional to the velocity.

A solution to this DE (whether explicit, numerical, or graphical) is only as valid as this
assumption. If the force due to air resistance behaves much differently than we think, then
our solution is invalid, regardless of the rigor of our mathematics.

**Simplicity and Proportionality**

One over-arching goal when creating mathematical models is to keep them simple. We often
start with the simplest possible model that has the potential to explain a given phenomena.
If the predictions of the model are not sufficiently accurate for the application, then a
more detailed model can be developed. This is essentially a variation of a philosophical
principle called *Occam’s Razor*, which states that among competing explanations of a given
phenomena, one should choose the simplest one.

One way to help keep a model simple is to simplify the description of the relationship
between two variables. *Proportionality* is often used to do just this.
1.3 Mathematical Modeling with Differential Equations: General Principles

**Definition 1.3.2** Two variables $x$ and $y$ are said to be directly proportional, or simply proportional, if there exists a constant $c \neq 0$, called the constant of proportionality such that

$$ y = cx. $$

The variables are said to be inversely proportional if

$$ y = \frac{c}{x}. $$

Two important observations need to be made:

1. If $x$ and $y$ are directly proportional, and $c > 0$, then if one variable increases, so does the other.

2. If $x$ and $y$ are inversely proportional, and $c > 0$, then if one variable increases, the other decreases.

In the crumpled paper scenario of Section 1.1, we modeled the force due to air resistance as proportional to the velocity. Hopefully this seems reasonable because if velocity increases, then the force due to air resistance will also increase. Granted, proportionality may not be the only way of modeling this relationship, but it is the simplest.

**Example 1.3.2** One famous law of nature involving inverse proportionality is Newton’s law of universal gravitation. This law relates the distances between two objects with the gravitational force between them. In mathematical terms, this law states

$$ F = G \frac{m_1 m_2}{r^2} $$

where $m_1$ and $m_2$ are the masses, $r$ is the distance between them, and $G$ is the proportionality constant, called the gravitational constant. In words, this law states that $F$ is inversely proportional to $r^2$, meaning that as the distance between two objects increases, the gravitational force decreases. This relationship agrees with our intuition.

Another well-known mathematical model involving proportionality is Newton’s law of cooling.
Example 1.3.3 (Newton’s law of cooling) Consider the following observation:

A hot cup of coffee sitting on a desk initially cools very quickly. As its temperature approaches room temperature, the coffee cools much slower.

This observation can be used to model the temperature of the coffee at any point in time. If we let \( T(t) \) represent the temperature at time \( t \) and \( A \) represent the ambient temperature (which we will assume is constant), then this observation suggests the following two relationships:

1. When the difference between the ambient temperature and the temperature of the coffee, \( (A - T) \), is large, then the change in \( T \), \( \frac{dT}{dt} \), is large.
2. When the difference is small, then the change is small.

These relationships suggest that \( \frac{dT}{dt} \) is directly proportional to \( (A - T) \). This yields the DE

\[
\frac{dT}{dt} = k(A - T)
\]

where \( k > 0 \) is a constant. This DE is known as Newton’s law of cooling. It can be used to describe the temperature of a hot object cooling, or a cool object warming.

If \( A \) and \( T \) are measured in °F and \( t \) is measured in minutes, then the units on the left side of the DE would be \( \frac{°F}{min} \) and the units of \( (A - T) \) would be °F. Thus to make the units on both sides of the DE agree, the units of \( k \) would need to be \( \text{min}^{-1} \). The constant \( k \) can be interpreted as a measure of how quickly the object cools (or warms).

Note that if \( A < T \), meaning that the object is warmer than ambient temperature, then \( (A - T) < 0 \). Thus \( \frac{dT}{dt} < 0 \) and the object’s temperature would be decreasing towards the ambient temperature. Conversely, if \( A > T \) then \( \frac{dT}{dt} > 0 \) so the object’s temperature would increase towards the ambient temperature. This observation agrees with our intuition.\(^4\)

Figure 1.9 shows a slope field along with several solution curves for the case \( A = 50°F, \ k = 0.7 \). As intended, objects starting out with temperature less than 50°F warm up until they reach 50°F, and objects that start out with temperature greater than 50°F cool until they reach 50°F.

\(^4\)Note that if we had constructed the model as \( \frac{dT}{dt} = k(T - A) \), then for this observation to be true we would need \( k < 0 \). This would complicate the model. We prefer the simpler option.
In some cases a proportionality relationship may seem reasonable, but lead to unreasonable prediction. This causes us to refine the model, as illustrated in the next example.

**Example 1.3.4** In Example 1.1.8 we modeled the growth of a population based on the assumption that the rate of growth is proportional to the size of the population. This yielded the basic DE model \( y' = ky \) where \( y(t) \) is the population at time \( t \) and \( k \) is a constant called the growth rate. This assumption means that large populations grow faster than small populations. This certainly seems reasonable. However, in the example, we also showed that if \( k = 3 \), then a general solution to this DE is \( y = Ce^{3t} \).

From calculus we know that if \( C > 0 \), then \( \lim_{t \to \infty} Ce^{3t} = \infty \), meaning that the population will grow **without bound**. This also means that the population will grow very quickly. These predictions do not seem reasonable. At some point the population will run out of food, space, or other resources and the rate of growth will slow down.

To refine the model, we want to describe the growth rate \( k \) as a quantity that decreases as \( y \) increases. The simplest way to do this is to describe \( k \) using a straight line with a negative slope. That is, we let \( k = a - by \) where \( a \) and \( b \) are positive constants. Substituting this
expression for $k$ into the original DE yields the new DE

\[ y' = (a - by)y. \]

Rewriting this DE we get

\[ y' = (a - by)y = ay \left(1 - \frac{y}{a/b}\right) = ay \left(1 - \frac{y}{N}\right) \]

where $N = a/b$. This DE is called a Logistic population model. Figure 1.10 shows a slope field for this DE with three solution curves using $a = 0.7$ and $N = 5$.

![Slope field for Logistic differential equation](image)

**Figure 1.10:** Logistic differential equation

From the slope field we see that $y(t)$ will approach the line $y = N$ regardless of the initial population. In population biology $N$ is called the carrying capacity, and represents the size of the population that can be sustained by the environment. This refined population model predicts constrained growth, which is much more reasonable than the unconstrained growth predicted by the original model.

The parameter $a$ is called the small population growth rate. This is because if $y$ is very small compared to the carrying capacity $N$, then the term $y/N$ is very small compared to 1 and hence $1 - y/N \approx 1$. The DE then becomes $y' = ay$, that is, the basic population
model, with growth rate $a$.  

**Balance of Units**

When creating a model, we must be careful about units as stated in the following principle:

The units on both sides of a DE must always agree.

This principle helps to explain the necessity of a constant of proportionality and its meaning as illustrated in the next example.

**Example 1.3.5** Suppose we tried to model a population of bacteria in a Petri disk with the DE $y' = y$ where $y$ represents the number of bacteria time is measured in hours. The units on the left-hand side would be bacteria per year and the units on the right would be bacteria. These units don’t agree. One way to solve this problem is to introduce a constant $k$ in the model to get $y' = ky$ where $k$ has the units $\text{hours}^{-1}$.

So what does this constant $k$ represent in a practical sense? We can rewrite this DE as

$$k = \frac{y'}{y}$$

which illustrates that $k$ represents the net growth rate of the population per unit of population.

To further illustrate this interpretation, suppose the population were increasing at a rate of 100 bacteria per hour when there are 1000 bacteria in the dish, then

$$k = \frac{100}{1000} = 0.10$$

and the units of $k$ would be

$$\frac{\text{bacteria/hour}}{\text{bacteria}} = \frac{1}{\text{hours}} = \text{hours}^{-1}$$

which are called “reciprocal hours.” What does this mean? Think in percentage terms. The value $k = 0.10$ indicates that each bacteria produces 0.10 bacteria per hour, meaning the bacteria are increasing at a rate of 10% per hour. Thus whatever the value of $k$, it represents a $100k\%$ net growth rate. It could also be interpreted as the difference between the birth rate and the death rate.
Approximating discrete variables with continuous ones

Variables representing quantities such as population can only take whole number values. Variables representing quantities such as mass do not have to take whole number values. This observation leads to the following definition.

**Definition 1.3.3** A variable is said to be *continuous* if it can take any value within some interval. A variable is *discrete* if its set of possible values is finite or countable.

Informally, a variable is discrete if there are “gaps” between consecutive values. A variable is continuous if there are no gaps. Often when modeling with DE’s we approximate discrete variables with continuous ones. This can greatly simplify our model because modeling discrete variables exactly can be very complicated.

In Example 1.3.5, the dependent variable \( y \) represents a population, which must be a whole number, so \( y \) is discrete. However, we described \( y \) with the DE \( y' = ky \). When we work with the derivative of \( y \), we are treating \( y \) as a continuous variable. In other words, we approximated a discrete variable with a continuous one.

We must keep this approximation in mind when we interpret the results of the model. If the model were to predict, say \( y(5) = 24.6 \), this would not mean there will be exactly 24.6 bacteria at time 5. We would interpret this as meaning there would be about 24 or 25 bacteria at time 5.

An approximation such as this tends to work best when the gaps between consecutive values of the dependent variable is small compared to possible values of the variable. For example, if time is our independent variable, and the population of the United States is our dependent variable, then the gaps between consecutive values is 1, which is rather small compared to populations in excess of 300 million.

Note that in Example 1.3.5, the independent variable representing time in hours is continuous. In DE’s, the independent variable must be continuous in order for the derivative of the unknown function to exist. In some practical applications, both the independent and dependent variables are discrete. The next example illustrates such a scenario.

**Example 1.3.6** The ring-tailed lemur has about a one month breeding season which runs from mid April to mid May, and the young are usually born in September. Let \( y_t \) represent
the number of lemurs in a population of lemurs at the beginning of year $i$, where $i = 0, 1, 2, \ldots$. Note that both the independent variable $i$ and dependent variable $y_i$ are discrete.

We can model this scenario with a \textit{discrete logistic model}

$$y_{i+1} = y_i + ay_i \left(1 - \frac{y_i}{N}\right)$$

where $a$ is a constant and $N$ is the carrying capacity. In words, this model says that the population in the next year, $y_{i+1}$, is equal to the population in the current year, $y_i$, plus a fraction of the current population. As the current population gets closer to the carrying capacity, this fraction gets closer to 0. Hopefully this model makes intuitive sense.

To illustrate how this model can be used to predict populations, suppose $N = 100$, $a = 0.4$, and $y_0 = 50$. Then $y_1$ is

$$y_1 = 50 + (0.4)(50)(1 - 50/100) = 60.0$$

and $y_2$ is

$$y_2 = 60 + (0.4)(60)(1 - 60/100) = 69.6.$$ 

We interpret this value of $y_2$ as meaning that there will be about 70 lemurs at the beginning of year 2. Other values of $y_i$ can be calculated in a similar fashion. Figure 1.11 (a) shows a graph of these values. In this figure, the points on the graph represent the actual values of $y_i$; the connecting lines are there just to make the graph easier to read. The applet corresponding to this example at \texttt{uhaweb.hartford.edu/rdecker/DeckerDEbook/DeckerDEbook.html} can be used to create similar graphs and can be experimented with interactively.

If we treat both the population and time as continuous variables, we can model this scenario with the \textit{continuous logistic model}

$$y' = ay \left(1 - \frac{y}{N}\right).$$

Figure 1.11 (b) shows the solution to this continuous model for $N = 100$, $a = 0.4$, and $y_0 = 50$ as approximated using Euler’s method with step size 0.01. Clearly for these parameter values, the two models make very similar predictions. However, when $N = 100$ and $a = 2.3$, things are quite a bit different as shown in Figure 1.12.

With the discrete model, we see the population oscillates back and forth between two values after about time period $i = 7$. In the continuous model the population reaches the carrying capacity and then levels out. We conclude that if the situation to be modeled is truly discrete, a continuous approximation may be very inaccurate, at least for certain
parameter values.

Models such as equation (1.3) are called difference equations, discrete dynamical systems, recurrence relations, or iterated maps. We will often just refer to them as discrete models. Equation (1.3) has the general form

\[ y_{i+1} = f(y_i). \]  

(1.4)

In words, this equation says that one value of \( y \), \( y_{i+1} \), is a function of the previous value of \( y \), \( y_i \). Now recall Euler’s method from Section 1.2 which was described with the equations

\[
\begin{align*}
x_{i+1} &= x_i + \Delta x \\
y_{i+1} &= y_i + f(x_i, y_i) \Delta x.
\end{align*}
\]

Note that the equation for \( y_{i+1} \) is of the same general form as equation (1.4). Thus Euler’s method is really a way of approximating a DE with a discrete model.

**Rate of accumulation**

Many scenarios involve a quantity that is being both increased and decreased. We can describe the overall rate of change of this quantity with a basic modeling principle. To motivate this principle, consider the following scenario:

If you are rolling bowling balls down a lane at a rate of 10 per minute, and
the balls are coming back to you at a rate of 8 per minute, then there is a pile of bowling balls somewhere growing larger at a rate of 2 per minute. Many physicists call this the “law of conservation of bowling balls,” though some insist on the more formal “law of conservation of matter.”

In more mathematical terms, we can describe this principle as

\[
\text{Rate of accumulation} = \text{Rate in} - \text{Rate out}.
\]

Our first application of this principle involves the idea of a concentration. The concentration of a material in a solution is defined as

\[
\text{Concentration} = \frac{\text{Total amount of material in the solution}}{\text{Total volume of the solution}}.
\]

The material could be a solid, such as salt, or it could be a liquid such as alcohol.

**Example 1.3.7** Consider a hot tub which is fed by a trickle of water from a hot spring, and water flows out of the tub through a small hole in the bottom. Now assume that the spring water coming into the hot tub contains a contaminate at the concentration of 2 grams per gal of water. If the hot tub initially contains 40 gal of pure water, water flows into the tub at the rate of 0.5 gal/min, water drains out of the tub at the same rate, and the mixture in the tub is continuously well-stirred, describe the amount of contaminate in the hot tub as a function of time.
Solution. Let \( y(t) \) represent the amount of contaminant (in grams) in the tub at time \( t \) (in min). Note that contaminant is accumulating in the tank. The rate of accumulation is rate at which the amount of contaminant in the tank is changing. In terms of a derivative, this rate is \( \frac{dy}{dt} \) and its units are g/min.

We can describe \( \frac{dy}{dt} \) using the rate of accumulation principle. By the balance of units principle, all the rates must have the same units, g/min. The Rate in is the rate at which contaminant is entering the tub. This is calculated by

\[
\text{Rate in} = \left( 0.5 \frac{\text{gal}}{\text{min}} \right) \cdot \left( 2 \frac{\text{g}}{\text{gal}} \right) = 1 \frac{\text{g}}{\text{min}}.
\]

Note that we calculated this rate by multiplying the rate at which solution flows into the tank by the concentration of the solution flowing in. The Rate out can be described the same way. Solution is flowing out at a rate of 0.5 gal/min. Because solution flows into the tub at the same rate it flows out, the volume of solution in the tub is a constant 40 gal. The concentration of the solution flowing out of the tub is

\[
\text{Concentration} = \frac{\text{Total amount of contaminant in the tub}}{\text{Total volume of solution in the tub}} = \frac{y}{40} \frac{\text{g}}{\text{gal}}.
\]

Thus the Rate out is

\[
\text{Rate in} = \left( 0.5 \frac{\text{gal}}{\text{min}} \right) \cdot \left( \frac{y}{40} \frac{\text{g}}{\text{gal}} \right) = \frac{y}{80} \frac{\text{g}}{\text{min}}.
\]

and the DE describing \( y \) is

\[
\frac{dy}{dt} = 1 - \frac{y}{80}.
\]

Since the tub initially contained pure water, we have the initial condition \( y(0) = 0 \). A slope field with with the specific solution corresponding to this initial condition is shown in Figure 1.13. From the slope field, we see that the amount of contaminant in the tub approaches 80 g in the long term, regardless of the initial amount of contaminant. Hopefully this makes sense. The solution in the tub is slowly being replaced by the solution from the hot spring which has a concentration of 2 g/gal. When the solution in the tub is totally replaced, there will be

\[
\left( 2 \frac{\text{g}}{\text{gal}} \right) \cdot (40 \text{gal}) = 80 \text{g}
\]

of contaminant in the tank.

The problem in this example is called a mixing problem. Generalizing this example, we see
that the DE describing \( y(t) \) has the form

\[
\frac{dy}{dt} = \left( \text{Rate of sol flowing in} \right) \cdot \left( \text{Conc of sol flowing in} \right) - \left( \text{Rate of sol flowing out} \right) \cdot \left( \frac{y}{\text{Vol of sol in tub}} \right).
\]

We will use this model in later sections.

**Exercises**

1.3.1 A decaying material, such as a radioactive material, is said to be in *exponential decay* if the rate at which the amount of material decreases is proportional to the amount of remaining material.

a. Let \( y(t) \) represent the amount of material remaining at time \( t \). Derive a DE to model \( y \). Use \( k > 0 \) as the constant of proportionality.

b. Verify that a general solution to the DE in part a. is \( y = Ce^{-kt} \) and show that \( C = y(0) \).

c. Suppose \( y \) is measured in grams (g) and \( t \) is measured in seconds (sec). Use the balance of units principle to find the units of \( k \).

d. The *half-life*, \( t_{\text{half-life}} \), of a material in exponential decay is the time required for \( y \) to be reduced to half of the original quantity. That is \( t_{\text{half-life}} \) is the time at which \( y(t) = 0.5y(0) \). Use the
general solution in part b. to show that
\[ t_{\text{half-life}} = -\frac{1}{k} \ln \left( \frac{1}{2} \right) \approx \frac{0.69315}{k}. \]

e. The radioactive isotope plutonium-239 has a half-life of about 24,000 years. If initially there is 1 g of plutonium-239, approximate the time at which there is 0.8 g remaining.

1.3.2 Use the model \( mv' = -cv|v| - mg \) for an object falling through a fluid medium (assuming turbulent flow) to estimate the time it will take a dropped crumpled piece of paper (initially zero velocity) to reach a velocity of \( \frac{2 \text{ meter}}{\text{sec}} \) in the downward direction. Assume that \( m \) is 0.0045 kilograms and that \( c = 0.009 \text{ newtons}\cdot\text{sec}^2/\text{meter}^2 \). Thus \( \frac{c}{m} = 2 \) so that the differential equation becomes \( v' = 2v^2 - 9.8 \). We have divided the differential equation by \( m \) and replaced \( v|v| \) with \( -v^2 \) because \( v \) is negative for a falling object. Use whatever method(s) you think are most appropriate for this problem (algebraic, numerical or graphical). Will the paper ever reach a velocity of \( \frac{3 \text{ meter}}{\text{sec}} \) in the downward direction? Explain.

1.3.3 For the model \( mv' = -cv|v|^{p-1} - mg \) of an object falling through a fluid medium, explain how the parameter \( p \) affects the graph of \( v \) versus \( t \). For simplicity assume that \( m = c = 1 \) and \( g = 9.8 \). Also assume that \( v(0) = 0 \) to represent an object dropped from rest. Recall that \( p = 1 \) represents laminar flow and \( p = 2 \) represents turbulent flow; consider values of \( p \) ranging between 0.1 and 5. Use whatever method(s) you think are most appropriate. Give an interpretation of your answer in terms of the falling object.

Note: when graphing the results of a numerical method, be sure to try different step sizes before you believe that the graph is accurate. A good rule of thumb for Euler’s method is to try two different step sizes with the second step size \( \frac{1}{10} \) of the first (such as 0.01 and 0.001); if the results are the same to a given number of digits, then those digits are accurate. For graphs, the graphs for the two different step sizes should appear to be (almost) the same. The theory behind this rule will be developed in Chapter 2.

1.4 Deriving Models of Electrical Circuits

The analysis of electrical circuits is a major application of differential equations. In order to dig into this area, we need some background in the physical laws of circuits. We will be primarily interested in circuits that contain only resistors, capacitors, inductors, and a voltage source. Such a circuit is called an \textit{RLC series circuit}. Figure 1.14 show a typical schematic for an RLC circuit. A capacitor, inductor, or resistor is generically called a \textit{component}.

An electrical current is a flow of electrons. A current flows through the circuit somewhat like water through pipes. A voltage source acts like a pump to push electrons through
1.4 Deriving Models of Electrical Circuits

Figure 1.14: RLC series circuit

the circuit. As the current passes through each component, there is a voltage drop. The relationships between the voltage drop across each component and the current through that component are used to derive differential equation(s) describing the circuit.

Series and parallel connections Components connected in series are connected along a single path, so the same amount of current flows through all of the components. Components are connected in parallel when the path splits into two or more paths with a component on each path, and then join up again (as when a river splits into two channels which later join back together).

Basic units A coulomb is a measure of electric charge consisting of about $6.24 \times 10^{18}$ elementary charges (a proton has an elementary charge of +1 and an electron has an elementary charge of −1). The letter $Q$ is typically used to represent an amount of charge. Current, denoted $I$, is the flow rate of charge with respect to time. In mathematical terms, $I$ and $Q$ are related by

$$I = \frac{dQ}{dt}.$$  

An ampere (abbreviated amp) of current is a flow of one coulomb per second. A volt is a measure of (potential) energy. If we think of an electrical circuit as the flow of water through a pipe, then very informally the current is a measure of how fast the water is flowing and the voltage is a measure of how much energy this flow has. Each component in the circuit changes the amount of energy in the flow.

Resistors A resistor is anything that reduces the flow rate of charge. That is, resistors
reduce the current. The relationship between the voltage drop $V$ across a resistor and the current $I$ through the resistor is given by Ohm’s law,

$$V = IR,$$

where $R$ is a constant called the *resistance*. By the balance of units principle, the unit of $R$ is \( \text{volt/amp} \). This unit is also called an *ohm*.

**Capacitors** Capacitors consist of at least two electrical conductors, generically called *plates*, separated by an insulator. Capacitors can store energy when a positive charge builds up on one plate and an equal negative charge builds up on the other. If the two sides are then connected by a conductor, current flows from one side to the other. Generally electrons flow from the negative side to the positive side, but for historical reasons, the current $I$ is considered to go the other way.

If $Q$ represents the charge on one plate of a capacitor (with an equal and opposite charge on the other plate), then the voltage drop $V$ across the capacitor is related to $Q$ by the equation $V = Q/C$ where $C$ is a constant called the *capacitance* of the capacitor. By the balance of units principle, the unit of $C$ is \( \text{coulomb/volt} \). This unit is also called a *farad*. Since $I = dQ/dt$, we can write the voltage-current relationship for a capacitor as

$$\frac{d}{dt} V = \frac{d}{dt} \left( \frac{Q}{C} \right) = \frac{dQ}{dt} \cdot \frac{1}{C} \Rightarrow \frac{dV}{dt} = \frac{I}{C}.$$

**Inductors** An inductor typically consists of a coiled wire. When electricity passes through the coil a magnetic field is created. A change in the current creates a corresponding change in the magnetic field which, in turn, generates an electromotive force (EMF) in the conductor that opposes this change in current. Thus inductors oppose changes in current through them. *Inductance* is a measure of the amount of EMF generated (in volts) per unit change in current.

The voltage drop across an inductor is related to the current according to the equation

$$V = L \frac{dI}{dt},$$

where $L$ is a constant measuring the inductance. By the balance of units principle, the unit of $L$ is \( \text{volt-second/ampere} \). This unit is also called a *henry*.

We summarize the voltage-current relationships for the three components in Table 1.3.

We use these voltage drop equations along with two rules, called Kirchhoff’s laws, to write the differential equation, or system of equations, corresponding to the circuit.
### Component Voltage drop

<table>
<thead>
<tr>
<th>Component</th>
<th>Voltage drop</th>
</tr>
</thead>
<tbody>
<tr>
<td>Resistor</td>
<td>$V = RI$</td>
</tr>
<tr>
<td>Capacitor</td>
<td>$V = Q/C$</td>
</tr>
<tr>
<td>Inductor</td>
<td>$V = L \frac{dI}{dt}$</td>
</tr>
</tbody>
</table>

**Table 1.3:** Voltage drops for RLC components

---

**Kirchhoff’s laws**

**Kirchhoff’s current law** At any node (junction) in an electrical circuit, the sum of currents flowing into that node is equal to the sum of currents flowing out of that node.

**Kirchhoff’s voltage law** The directed sum of the voltage drops around any closed loop is zero.

---

**Turning circuits into differential equations** Given a circuit, label each component with the current through that component, such as $I_1$, $I_2$, and so on. Also, give a direction for each current (this is because you need the “directed” sum of voltages in Kirchhoff’s voltage law). Use Kirchoff’s current law to eliminate as many of the currents $I_1$, $I_2$, ... as possible (get to a minimal number of independent currents).

Next apply Kirchhoff’s voltage law and the voltage drop rules from Table 1.3 to as many closed loops as is necessary to write the differential equation(s). When voltage sources are present, the voltage “drop” is just the voltage of the source. For a DC circuit the voltage source would be constant (such as $V(t) = 12$), and for AC circuits voltage sources are typically sinusoidal (such as $V(t) = 3 \sin(2\pi t)$). The voltage drop for a source should have the opposite sign to those of the components (resistor, capacitor, inductor) because the voltage increases rather than dropping. Note: A “loop” is any closed path in the circuit, so there are often many more loops present than are necessary to create all the needed DE’s; the others will be redundant.

**Example 1.4.1** Consider the RLC series circuit shown in Figure 1.14. At each node (the black dots that separate the components) we apply Kirchhoff’s current law, and see that the current through each component is the same, which we denote $I$ (the current into each node is $I$ so the current out must also be $I$). There is only one closed loop (the entire
circuit), so by Kirchhoff’s voltage law the sum of the voltage drops around the circuit must be zero. Using Table 1.3 to calculate the voltage drops we get \( L \frac{dI}{dt} + RI + \frac{Q}{C} - V(t) = 0 \) where \( V(t) \) is the voltage source (and so has a negative sign). Differentiating this equation with respect to time and rearranging we get

\[
L \frac{d^2I}{dt^2} + R \frac{dI}{dt} + \frac{1}{C} I = V'(t)
\]

(RLC series circuit)

since \( \frac{dQ}{dt} = I \).

Example 1.4.1 illustrates the basic RLC series circuit; understanding this circuit is critical to understanding larger circuits. If either the inductor or capacitor are eliminated we get important first-order equations. If the inductor is eliminated (equivalent to setting \( L = 0 \)) we obtain the \( RC \) circuit equation

\[
R \frac{dI}{dt} + \frac{1}{C} I = V'(t)
\]

(RC series circuit)

If the capacitor is eliminated (equivalent to setting \( C = \infty \)) we get the \( RL \) circuit equation

\[
L \frac{dI}{dt} + RI = V(t)
\]

(RL series circuit)

(for \( RL \) circuits we integrate once in order to get a first-order DE).

Input-output interpretation  If we consider the voltage source \( V(t) \) to be the input, then we can pick one of the components in the circuit (say a resistor) and call the voltage drop across the resistor the output. Thus, given \( V(t) \) we can solve the differential equation (or DE system) for the various currents \( I_1, I_2, ... \) and then calculate the output voltage using the appropriate voltage-current relationship from Table 1.3.

We finish our discussion of circuits with an \( RLC \) circuit that has more than one loop.

Example 1.4.2 Consider the circuit in Figure 1.15. Find a system of differential equations that describes this circuit.

Solution. Based on the labeling of the currents \( i, i_1, i_2 \) in Figure 1.15 we can use Kirchhoff’s current law to get \( i = i_1 + i_2 \) (using either of the two nodes). There are three loops to which we can apply Kirchhoff’s voltage law: the left loop, the right loop, and the loop around the outside. For the loop around the outside we get \( R i + R_1 i_1 + \frac{1}{C_1} q_1 + \frac{1}{C} q = V(t) \) where \( q \) and \( q_1 \) are the charges on the capacitors (so that \( \frac{dq}{dt} = i \) and \( \frac{dQ_1}{dt} = i_1 \)). For the loop on
Using Lagrange’s Equations to Model Mechanical Systems

We have used Newton’s second law $\sum F = ma$ to create differential equations for some simple mechanical systems. There is a different approach developed by Joseph-Louis Lagrange, which is very useful for developing the equations of motion for more complex systems, especially when constraints are involved\(^5\). This approach requires first calculating the kinetic

\[ R \left( \frac{d^2i_1}{dt^2} + \frac{d^2i_2}{dt^2} \right) + R_1 \frac{di_1}{dt} + \frac{1}{C_1} i_1 + \frac{1}{C} (i_1 + i_2) = V'(t) \]  

(1.5)

and

\[ L \frac{d^2i_2}{dt^2} - R_1 \frac{di_1}{dt} - \frac{1}{C_1} i_1 = 0 \]  

(1.6)

Note that if we use the loop on the left we get the equation $Ri + L \frac{di_2}{dt} + \frac{1}{C} q = V(t)$ which is just Equation 1.6 added to Equation 1.5, and hence gives no additional information. We will be able to find an explicit solution to this system by writing it as a system of three first-order equations when we get to Chapter 5.

\( ^5 \)We will not provide much of the theory behind this approach; for a full development see [1] Landau, L.D.
and potential energies of the entire system, and then using the Euler-Lagrange equations to derive the equations of motion.

In order to apply Lagrange’s approach, we need a little bit of background in dynamics. First we distinguish between particles and rigid bodies. Particles are also referred to as point masses; you can think of a particle as a sphere with zero radius (and hence is zero dimensional) and positive mass. A rigid body on the other hand is a one, two, or three dimensional object with mass, which cannot be deformed (the distances between points in the body remain the same). For many calculations, a rigid body of mass \( m \) can be considered to be a particle of mass \( m \), with the mass concentrated at the center of mass of the body. The exception is that for rotating rigid bodies, the kinetic energy must take into account both the translational and rotational motions.

**Kinetic energy** For a particle of mass \( m \) and velocity \( v \) in one dimensional motion (such as a mass in free fall or attached to a spring and damper as discussed previously), the kinetic energy is given by \( T = \frac{1}{2}mv^2 \). For motion in three dimensions, where the velocity is a vector, then \( T = \frac{1}{2}mv \cdot v \) (recall that the dot product of the vectors \( v = v_x \hat{i} + v_y \hat{j} + v_z \hat{k} \) and \( w = w_x \hat{i} + w_y \hat{j} + w_z \hat{k} \) is given by \( v \cdot w = v_xw_x + v_yw_y + v_zw_z \)). Finally, for a system of \( N \) particles of masses \( m_1, m_2, \ldots, m_N \) with velocities \( v_1, v_2, \ldots, v_N \) the kinetic energy is just the sum of the energies of the individual particles, that is \( T = \sum_{i=1}^{N} mv_i \cdot v_i \).

For rigid bodies, we calculate the translational kinetic energy by considering the mass to be concentrated at the center of mass, and using the rules for particles. For motion in a plane (we will not consider the complexities of general three dimensional motion in this text), if the object is rotating, with its axis of rotation perpendicular to the plane, we can then calculate its rotational kinetic energy using \( \frac{1}{2}I\omega^2 \). Here \( \omega \) is the rotational velocity (in radians per time unit) and \( I \) is the moment of inertia about the center of mass of the rigid body. \( I \) can be calculated using \( I = \int \int r^2 \rho dV \) (\( \rho \) is the density and \( r \) is the distance to the center of mass) or can be looked up in tables for standard shapes.

**Potential energy** Certain forces, called conservative forces, can be derived from a potential energy function. The forces of gravity and of a linear spring are conservative. Friction is an example of a nonconservative force. If the force on a particle is conservative, and it depends on a position variable \( x \), we can denote the force by \( F(x) \). The potential energy of the particle, denoted \( V(x) \), is then related to the force by the equation \( F(x) = -V'(x) \)

Using Lagrange’s Equations to Model Mechanical Systems

(in higher dimensions we replace the derivative of $V$ with the gradient of $V$). Conservative systems conserve energy and nonconservative systems dissipate (lose) energy.

Two forms of potential energy that we will be concerned with are those of a constant gravitational field and of a (linear) spring. The potential energy of an object of mass $m$ that is a distance $h$ above a given reference point (which is arbitrary) in a constant gravitational field with acceleration $g$ is given by $V(h) = mgh$. Thus the corresponding force is $F(h) = -V'(h) = -mg$ as we have used before. For a spring with spring constant $k$ the potential energy when stretched a distance $x$ is $V(x) = \frac{1}{2}kx^2$; thus the force is $F(x) = -kx$ as before.

The Euler-Lagrange equations For a given mechanical system, we find a smallest set of independent variables $q_1, q_2, \ldots, q_N$ (called generalized coordinates) required to describe the configuration of the system, without having to add additional algebraic equations (called constraints). We then determine the kinetic ($T$) and potential ($V$) energies of the system as a function of those variables and their derivatives $q_i', q_2', \ldots, q_N'$. Next form a quantity called the Lagrangian, given by $L = T - V$. The Euler-Lagrange equations are

$$\frac{d}{dt} \frac{\partial L}{\partial q_i'} = \frac{\partial L}{\partial q_i} \quad \text{for } i = 1, 2, \ldots, N.$$  

(Euler-Lagrange equations)

For the purpose of taking the partial derivatives, the variables $q_i$ and $q_i'$ are considered to be independent. The Euler-Lagrange equations result in a set of differential equations of motion that are equivalent to those derived using Newton’s second law.

Note: While it is possible to include nonconservative forces (such as friction) in the Lagrangian approach, in this text we will use this approach only for conservative systems. In some cases, after deriving the equations of motion for a conservative system, we can adjust those equations to account for friction.

Lagrangian method for developing equations of motion for conservative mechanical systems

1. Find the kinetic energy $T$ and potential energy $V$ in terms of cartesian coordinates.
2. Determine generalized coordinates $q_1, q_2, \ldots$ which describe the system without having to use constraint equations.

3. Write $T$ and $V$ in terms of the generalized coordinates and then form the Lagrangian $L = T - V$.

4. Apply the Euler-Lagrange equations to each generalized coordinate to find the differential equations of motion.

Before proceeding on to find the differential equations of motion for some systems that we have not encountered yet, let’s first check our new method by applying it to a system for which we already know the equations of motion. Recall the mass-spring-damper system consisting of an object of mass $m$, attached to a fixed object (such as a wall) by a spring with spring constant $k$, and including a damper with damping constant $c$. First, eliminate the damper, so that the system is conservative. The variable $x$ represents the amount of stretch of the spring and therefore completely describes the system (we have just one generalized coordinate). The potential energy is $V = \frac{1}{2}kx^2$ and the kinetic energy is $T = \frac{1}{2}mv^2 = \frac{1}{2}m(x')^2$. The Lagrangian is $L = \frac{1}{2}m(x')^2 - \frac{1}{2}kx^2$. Then $\frac{\partial L}{\partial x} = -kx$, $\frac{\partial L}{\partial x'} = mx'$, and $\frac{d}{dt} \frac{\partial L}{\partial x'} = mxx''$. The Euler-Lagrange equations become $mxx'' = -kx$ or $mxx'' + kx = 0$, which agree with our previous results when $c = 0$.

**Example 1.5.1** Consider a rigid pendulum which is constrained to move in a plane. We model the rigid pendulum as a particle of mass $m$ at the end of a massless rigid arm of length $l$. We use the variable $\theta$ to measure the angle of the arm with the vertical, with $\theta = 0$ representing the pendulum hanging straight down. See Figure . Find the equation of motion of the pendulum in terms of $\theta(t)$.

**Solution.** The system is completely described by the angle $\theta$ so we only need one generalized coordinate. The kinetic energy of the system is $T = \frac{1}{2}m(v_x^2 + v_y^2) = \frac{1}{2}m((x')^2 + (y')^2)$ where $x$ and $y$ are the cartesian coordinates of the mass. We can determine $x$ and $y$ in terms of $\theta$ using the polar coordinates $x = l \sin \theta$ and $y = -l \cos \theta$ (note that these are not the usual definitions of polar coordinates, but are correct for our definition of $\theta$). Thus $x' = l \cos(\theta)\theta'$ and $y' = l \sin(\theta)\theta'$ resulting in $T = \frac{1}{2}m((-l \sin \theta)\theta')^2 + (l \cos \theta)\theta')^2) = \frac{1}{2}ml^2(\theta')^2$ (basic trig identity). For the potential energy we have only the potential energy of a constant gravity field so that $V = mgh = -mgl \cos \theta$ where we have chosen the reference point to be the pivot point of the pendulum (and hence $h = -l \cos \theta$ is negative when $\theta = 0$ and positive when $\theta = \pi$).
Using Lagrange’s Equations to Model Mechanical Systems

Figure 1.16: A rigid pendulum

We now have $L = T - V = \frac{1}{2} ml^2 (\theta')^2 + mgl \cos \theta$ and hence $\frac{\partial L}{\partial \dot{\theta}} = -mgl \sin \theta$ and $\frac{\partial L}{\partial \theta'} = ml \dot{\theta}'$. Finally, $\frac{d}{dt} \frac{\partial L}{\partial \theta'} = ml \ddot{\theta}''$. The Euler-Lagrange equations yield $ml \ddot{\theta}'' = -mgl \sin \theta$ which after some simplifying and rearranging gives

$$\ddot{\theta}'' + \frac{g}{l} \sin \theta = 0.$$  
(Pendulum equation)

For physics enthusiasts: the first form of the of the pendulum equation $ml \ddot{\theta}'' = -mgl \sin \theta$ can be derived by balancing torque (the units of torque come from $\text{torque} = \text{force} \times \text{distance}$ or in MKS units newton-meters = kg·meters·sec$^{-2}$·meters). If this equation is divided by $l$ we get $ml \ddot{\theta}'' = -mg \sin \theta$ which can be derived using a balance of forces (Newton’s second law) in the direction of motion of the mass. See the Wikipedia article at en.wikipedia.org/wiki/Pendulum\_(mathematics) for derivations of these equations.

Notice that the similarity of the forms of the undamped mass-spring equation ($x'' + \frac{k}{m} x = 0$) and the pendulum equation ($\theta'' + \frac{g}{l} \sin \theta = 0$). A well-known property of the sine function is that for small $\theta$, $\sin \theta \approx \theta$; thus for small $\theta$ a pendulum behaves just as a linear spring. Thus, if we want to add damping to the pendulum equation, by analogy with the mass-spring equations, a reasonable form would be

$$ml \ddot{\theta}'' + c \dot{\theta}' + mg \sin \theta = 0.$$  
(Damped pendulum equation)

Notice that we have used the “balance of forces” form of the pendulum equation to tamper
with. Also, the reason for using $cl$ for the damping constant instead of just $c$ is to that $l\theta'$ becomes a velocity in analogy with the mass-spring equation (from a mathematical point of view there is no difference as the constant $cl$ could always be renamed as a single constant)\(^6\).

The real fun with this method comes in deriving equations of motion for more complicated (and interesting) mechanical systems, such as systems of connected springs and/or pendulums. A simulation can then be created by solving the resulting equations (or systems of equations) numerically.

**Example 1.5.2** Consider a “spring” pendulum consisting of mass attached to a pivot point by a massless spring. Assume the mass to be $m$ kg and the spring to have an unstretched length of $l$ meters. Let $z$ represent the amount of displacement of the spring from its unstretched length, so that the total length of the spring is $z + l$. See Figure 1.17. Find the equations of motion in terms of the angle $\theta$ and the displacement $z$.

**Solution.** Let $x$ and $y$ represent the cartesian coordinates of the mass relative to the pivot point. Thus $x = (l + z)\sin \theta$ and $y = -(l + z)\cos \theta$. The kinetic energy is then $T = \frac{1}{2}m(x'^2 + y'^2)$ and the potential energy is $V = mg y + \frac{1}{2}kz^2$ (we get potential energy both from a constant gravitational field and from the displacement of a spring). Now we to convert completely to the (generalized) coordinates $z$ and $\theta$. Taking derivatives we have $x' = (l + z)'\sin \theta + (l + z)(\cos \theta)\theta' = z'\sin \theta + (l + z)(\cos \theta)\theta'$ and similarly $y' = -z'\cos \theta + (l + z)(\sin \theta)\theta'$. This gives $T = \frac{1}{2}m((l'\sin \theta + l(\cos \theta)\theta')^2 + (-l'\cos \theta + l(\sin \theta)\theta')^2)$ and then after multiplying out and using the basic trig identity $\sin^2 \theta + \cos^2 \theta = 1$ twice we end up with $T = \frac{1}{2} mz'^2 + \frac{1}{2}m(z + l)^2 \theta'^2$. The potential energy becomes $V = -mg(l+z)\cos \theta + \frac{1}{2}kz^2$ and so the Langrangian is

\[
L = \frac{1}{2} mz'^2 + \frac{1}{2}m(z + l)^2 \theta'^2 + mg(l + z)\cos \theta - \frac{1}{2}kz^2
\]

\(^6\)See the article by John Hubbard at [www.math.cornell.edu/~hubbard/pendulum.pdf](http://www.math.cornell.edu/~hubbard/pendulum.pdf) for an article about the forced, damped pendulum that is a bit above the level of this text, but which the reader may find inspirational.
The appropriate derivatives are

\[
\frac{\partial L}{\partial z'} = mz' \\
\frac{\partial L}{\partial z} = m (z + l) \theta^2 + mg \cos \theta - k z \\
\frac{\partial L}{\partial \theta'} = m (z + l)^2 \theta' \\
\frac{\partial L}{\partial \theta} = -mg(l + z) \sin \theta
\]

and so the Euler-Lagrange equations give

\[
mz'' = m (z + l) \theta^2 + mg \cos \theta - k z \\
2m(z + l)z'\theta' + m (z + l)^2 \theta'' = -mg(z + l) \sin \theta
\]

This system of coupled second-order equations does not have an explicit solution in terms of elementary functions, but can be solved numerically after rewriting as a system of four first-order equations (see Chapter 5).
Chapter 2

First-order Differential Equations

In this chapter we delve deeper into the various types of solutions (explicit, numerical, and graphical) to first-order DE’s. In particular, we begin to develop methods for finding explicit solutions. These methods depend highly on the form of the DE; different forms require different methods. There is no “one size fits all” method for finding explicit solutions. In this text we discuss only a small number of the methods available to mathematicians for finding explicit solutions. More advanced texts discuss many additional methods.

In the first part of this chapter we discuss linear first-order equations. The distinction between linear and nonlinear differential equations is perhaps the most important that we will make throughout this entire text. In general, linear DE’s of all orders are much better understood than nonlinear ones, are usually easier to solve, and have solutions whose properties are often much “nicer” than those of nonlinear equations. The analysis of first-order nonlinear equations is the focus of the rest of this chapter.

2.1 Linear first-order differential equations

In high school algebra, a linear function of a single variable $x$ is a function of the form

$$f(x) = ax + b$$

where $a$ and $b$ are constants. Any function that cannot be written in the form is called nonlinear. For instance, the functions

$$f(x) = x^2 + 3, \quad f(x) = \frac{1}{x + 3}, \quad \text{and} \quad f(x) = \cos(x)$$

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are all nonlinear. A linear differential equation is a DE that can be written in a certain form. Linear DE’s can be of any order. In this section we deal with only first-order equations.

**Definition 2.1.1** A first-order differential equation is said to be linear if it can be written in the form

$$y' + g(x)y = h(x)$$

where $g(x)$ and $h(x)$ are functions of the independent variable. This form is called the standard form of a first-order linear DE.

Several comments can help clarify this definition:

1. The variables $y$ and $x$ are generic names for the dependent and independent variables, respectively. Any letter can be used for the names of these variables.
2. Parameters can appear in the functions $g(x)$ and $h(x)$. Also, these functions can be constant functions. For example, in the DE $y' + y = 2$ we have $g(x) = 1$ and $h(x) = 2$.
3. The functions $g(x)$ and $h(x)$ need not be linear functions of $x$.
4. A linear DE is often more easily identified by what it does not contain. A linear DE cannot contain the dependent variable, or any of its derivatives, raised to a power, in an exponent, in a denominator, or inside a trig function. Nor can the dependent variable, or any of its derivatives, be multiplied by any of its derivatives.

To determine if a given DE is linear or not, we need to determine if we can algebraically rewrite the equation into standard form. If we can, then the DE is linear. If we can’t, then it’s not linear.

**Example 2.1.1** Determine whether each of the following DE’s is linear or not. If it is linear, identify the functions $g$ and $h$.

a. $y' + 2y = 3x + 2$  
   b. $y' + 2y^2 = e^x$  
   c. $y' = 2y$  
   d. $y' = \cos(x)y + \frac{1}{x^2 + 1}$  
   e. $y' = x^2$  
   f. $x' + t^2 x = e^t$

**Solution.**

a. This DE is already in standard form where $g(x)$ is the constant function 2 and $h(x) = 3x + 2$. 
b. This DE is not linear because of the $y^2$ term.

c. We can rewrite this DE as $y' - 2y = 0$, so it is linear with $g(x) = -2$ and $h(x) = 0$.

d. We can rewrite this DE as $y' - \cos(x)y = 1/(x^2 + 1)$, so it is linear with $g(x) = \cos(x)$ and $h(x) = 1/(x^2 + 1)$. Notice that $g$ and $h$ are not linear functions of $x$. However, this is not important.

e. This DE contains the product of $y$ and $y'$. Thus it is not linear. We might be tempted to try to rewrite it into the standard form, but we will be unsuccessful.

f. In this differential equation, the dependent variable is $x$ and the independent variable is $t$. Note that the coefficient of the dependent variable is a function of the independent variable and that the right-hand side of the DE is a function of the independent variable. Thus this DE is linear with $g(t) = t^2$ and $h(t) = e^t$.

One reason that first-order DE’s are nice to work with is that there is a standard method of finding explicit solutions. To motivate this method, recall the product rule for derivatives from calculus:

$$\frac{d}{dx}(f(x)g(x)) = f'g + fg'.$$

**Example 2.1.2** To illustrate how the product rule can be used to find an explicit solution, consider the DE $xy' + y = 1$. This equation could be rewritten into the form of a linear DE (the reader should verify this), but this is not necessary. Observe that by the product rule,

$$\frac{d}{dx}(xy) = 1y + xy' = xy' + y$$

so that the DE can be written as

$$\frac{d}{dx}(xy) = 1.$$

Integrating both sides with respect to $x$ gives

$$\int \frac{d}{dx}(xy) \, dx = \int 1 \, dx \quad \Rightarrow \quad xy = x + C$$

where $C$ is an arbitrary constant. Solving for $y$ yields

$$y = 1 + \frac{C}{x}$$

which is a general solution.
CHAPTER 2  First-order Differential Equations

Not all linear DE’s are as simple as the previous example. To solve more complicated examples, recall the general exponential rule from calculus:

\[
\frac{d}{dx} e^{f(x)} = e^{f(x)} f'(x)
\]

where \( f(x) \) is a function of \( x \). Combining the product and exponential rules, we get the following formula:

\[
\frac{d}{dx} \left( ye^{f(x)} \right) = y' e^{f(x)} + ye^{f(x)} f'(x)
\]  

where \( y \) is a function of \( x \). The next example illustrates how this formula can be used to solve linear DE’s.

**Example 2.1.3** Consider the linear first-order DE \( y' + 2y = 3 \). Observe that the left-hand side of this DE resembles the right-hand side of equation (2.1), but without the exponential terms. To make these look more similar, we calculate the following quantity:

\[
\mu(x) = e^{\int 2 \, dx} = e^{2x}.
\]

Then we multiply both sides of the DE by \( \mu = e^{2x} \):

\[
e^{2x} \left( y' + 2y \right) = 3e^{2x} \quad \Rightarrow \quad y' e^{2x} + 2ye^{2x} = 3e^{2x}.
\]

Next comes the key step. Comparing the left-hand side of this equivalent version of the DE to the right-hand side of equation (2.1), we see that the DE is equivalent to

\[
\frac{d}{dx} \left( ye^{2x} \right) = 3e^{2x}.
\]

Integrating both sides with respect to \( x \) yields

\[
\int \frac{d}{dx} \left( ye^{2x} \right) \, dx = \int 3e^{2x} \, dx \quad \Rightarrow \quad ye^{2x} = \frac{3}{2} e^{2x} + C
\]

where \( C \) is an arbitrary constant. Solving for \( y \) yields

\[
y = \frac{3}{2} + Ce^{-2x}.
\]

This is a general solution to the DE.

The function \( \mu(x) \) calculated in this example is called an *integrating factor*. We generalize this example in the *integrating factor method* for solving first-order linear DE’s.
The Integrating Factor Method

Purpose: To find an explicit solution to a first-order linear DE.

1. Put the equation in the standard form $y' + g(x)y = h(x)$, if it is not already.
2. Calculate the integrating factor
   \[ \mu(x) = e^{\int g(x) \, dx}. \]

3. Multiply both sides of the DE by $\mu(x)$. The result can be rewritten in the form
   \[ \frac{d}{dx}[y\mu(x)] = h(x)\mu(x). \]

4. Integrate both sides of the DE in step 3 with respect to $x$ to get
   \[ y\mu(x) = \int h(x)\mu(x) \, dx + C. \]

5. Solve the equation in step 4 for $y$ to get
   \[ y = \mu^{-1} \int h\mu \, dx + C\mu^{-1}. \]

In step 2, $\int g(x) \, dx$ represents any antiderivative of $g(x)$, even though we leave off $+C$ for simplicity. Also, in step 5, $\mu^{-1}$ mean $1/\mu$, not the inverse function of $\mu$. Finally, we could simply memorize the final formula in step 5. However, we don’t recommend this. We recommend going through the steps each time, as they are not difficult and they help us remember why the method works.

Example 2.1.4 Use the integrating factor method to solve the DE $y' + y \cos(t) = \cos(t)$.

Solution. We follow the 5 steps outlined above:

Step 1: This DE is already in standard form with $t$ as the independent variable and $g(t) = \cos(t)$. 
Step 2: $\mu = e^{\int \cos(t) dt} = e^{\sin(t)}$

Step 3: 
\[ y' e^{\sin(t)} + y \cos(t) e^{\sin(t)} = \cos(t) e^{\sin(t)} \Rightarrow \frac{d}{dt} (ye^{\sin(t)}) = \cos(t) e^{\sin(t)} \]

Step 4: 
\[ \int \frac{d}{dt} (ye^{\sin(t)}) dt = \int \cos(t) e^{\sin(t)} dt \Rightarrow ye^{\sin(t)} = e^{\sin(t)} + C \]

Step 5: 
\[ y = e^{-\sin(t)} e^{\sin(t)} + Ce^{-\sin(t)} = 1 + Ce^{-\sin(t)} \]

Therefore, a general solution is $y = 1 + Ce^{-\sin(t)}$.

In the previous example, the integrating factor method was relatively easy to apply. However, in other cases the integrals $\int g(x) dx$ or $\int h(x) \mu(x) dx$ cannot be calculated in terms of elementary functions. In these cases the method fails to give an explicit solution. The next example illustrates one such case.

**Example 2.1.5** Try to use the integrating factor method to solve the DE $y' + y = \sqrt{x}$.

**Solution.** We try to follow the 5 steps:

Step 1: This DE is already in standard form with $x$ as the independent variable and $g(x) = 1$.

Step 2: $\mu = e^{\int 1 dx} = e^{x}$

Step 3: 
\[ y' e^{x} + ye^{x} = \sqrt{x} e^{x} \Rightarrow \frac{d}{dx} (ye^{x}) = \sqrt{x} e^{x} \]

Step 4: 
\[ \int \frac{d}{dx} (ye^{x}) dx = \int \sqrt{x} e^{x} dx \Rightarrow ye^{x} = \int \sqrt{x} e^{x} dx \]

The integral on the right in step 4 cannot be evaluated in terms of elementary functions (some computer algebra systems return a result in terms of the special function erf, some in terms of the special function erfi, and some simply return the integral itself indicating that an antiderivative cannot not be found). Thus the method fails to give an explicit solution. We could write the solution as $y = e^{-x} \int \sqrt{x} e^{x} dx + Ce^{-x}$ but this is not an explicit solution in terms of elementary functions.

**Applications**

The integrating factor method can be used to find the explicit solutions of real-world problems as illustrated in the next four examples.
Example 2.1.6 Recall Example 1.2.5 where a bacteria population was controlled with the continuous addition of an antibiotic. The resulting population was described by the DE

\[ y' = 3y - 100t. \]

Suppose the initial population is \( y(0) = 10 \). Based on our graphical analysis in Example 1.2.5, we predict that the population will eventually die out. Figure 2.1 shows the slope field for this differential equation along with the approximate solution curve for this initial population.

![Slope field for \( y' = 3y - 100t \)](image)

One question we might ask is “when will the population die out?” The solution curve in Figure 2.1 indicates the population will reach 0 at approximately \( t = 1.3 \). Also in Example 1.2.5 we concluded that the population will increases without bound for an initial population of approximately 11.11 or greater, and decrease to 0 for smaller initial populations. Find the explicit solution to this differential equation and verify these approximations.

**Solution.** To find the explicit solution, we follow the 5 steps in the integrating factor method:

Step 1: In standard form, the DE is \( y' - 3y = -100t \).

Step 2: \( \mu = e^{\int -3dt} = -3t \)

Step 3: \( y'e^{-3t} - 3ye^{-3t} = -100te^{-3t} \implies \frac{d}{dt} (ye^{-3t}) = -100te^{-3t} \)
Step 4: \[ \int \frac{d}{dt} (ye^{-3t}) \, dt = \int -100te^{-3t} \, dt \quad \Rightarrow \quad ye^{-3t} = \frac{100}{9}e^{-3t} + \frac{100}{3}te^{-3t} + C \] (the integral on the right is evaluated using integration by parts or with software)

Step 5: \[ y = \frac{100}{9} + \frac{100}{3}t + Ce^{3t} \]

If \( y(0) = 10 \), then

\[ 10 = \frac{100}{9} + \frac{100}{3}(0) + Ce^{3(0)} = C + \frac{100}{9} \quad \Rightarrow \quad C = -\frac{10}{9} \]

so that the particular solution is

\[ y = \frac{100}{9} + \frac{100}{3}t - \frac{10}{9}e^{3t} \]

To find the exact time at which the population dies out, we set \( y \) equal to 0 and solve for \( t \). The resulting equation cannot be solved algebraically, but it can be solved numerically using software. Software gives the solutions \( t = 1.29657 \) and \( t = -0.32059 \). The negative value of \( t \) makes no sense, so we ignore this solution. This confirms our graphical approximation of \( t \).

To analyze how the initial condition affects the long-term behavior of the solution, we examine the general solution found in Step 5. Note that there is a linear part, \( \frac{100}{9} + \frac{100}{3}t \), which is positive for all \( t > 0 \). There is also an exponential part, \( Ce^{3t} \) whose sign depends on the sign of \( C \). Thus the long-term behavior of the solution depends on the sign of \( C \). We analyze these three cases separately:

1. If \( C = 0 \), then the solution is \( y = \frac{100}{9} + \frac{100}{3}t \), whose graph is a straight line. The initial population would be \( y(0) = 100/9 \approx 11.11 \).
2. If \( C > 0 \), then \( y(t) > y(0) \) for all \( t \). Thus the population would be continually increasing.
3. If \( C < 0 \), then “eventually” the negative exponential part \( Ce^{3t} \) would dominate the positive linear part causing \( y(t) \) to become negative, meaning the population dies out.

This algebraic analysis confirms our graphical analysis. Figure 2.2 illustrates these three cases.

Admittedly, in this situation the additional accuracy gained by the algebraic analysis may not be terribly useful. The point here is that an algebraic analysis of a general solution gives higher levels of accuracy than other methods.
Example 2.1.7 In section 1.3 we derived Newton’s law of cooling,

\[
\frac{dT}{dt} = k(A - T)
\]

where \(A\) is the constant room temperature, \(k\) is a constant, \(T\) is the temperature of the object, and \(t\) is time.

Suppose that a pizza stone is removed from an oven at 375\(^\circ\)F and set on a table in a room at 70\(^\circ\)F. After 1/4 of an hour, the temperature of the stone is 150\(^\circ\)F. Using Newton’s law of cooling, find the exact time at which the stone reaches 90\(^\circ\)F.

Solution. First we find a general solution to Newton’s law of cooling using the integrating factor method. Rather than substituting \(A = 70\) into the differential equation at the beginning, we choose to leave \(A\) as a parameter and get the general solution in terms of \(A\). This gives us a result that we can reuse in other situations.

Step 1: \(T' + kT = kA\).

Step 2: \(\mu = e^{\int k\,dt} = e^{kt}\)

Step 3: \(T'e^{kt} + kTe^{kt} = kAe^{kt}\) \(\Rightarrow\) \(\frac{d}{dt}(Te^{kt}) = kAe^{kt}\)

Step 4: \(\int \frac{d}{dt}(Te^{kt})\,dt = \int kAe^{kt}\,dt\) \(\Rightarrow\) \(Te^{kt} = Ae^{kt} + C\)
Step 5: \( T = A + C e^{-kt} \)

We now substitute in \( A = 70 \) to obtain the general solution \( T = 70 + C e^{-kt} \). We still must determine the integration constant \( C \) and proportionality constant \( k \). To determine \( k \), we summarize the information given in the problem in the form of a table:

<table>
<thead>
<tr>
<th>( t )</th>
<th>( 0 )</th>
<th>( 0.25 )</th>
<th>?</th>
</tr>
</thead>
<tbody>
<tr>
<td>( T )</td>
<td>375</td>
<td>150</td>
<td>90</td>
</tr>
</tbody>
</table>

The first data point gives

\[ 375 = 70 + C e^{-k(0)} \quad \Rightarrow \quad C = 305. \]

Using this value of \( C \) and the second data point we get

\[ 150 = 70 + 305 e^{-k(0.25)} \quad \Rightarrow \quad k = \frac{1}{0.25} \ln\left(\frac{80}{305}\right) \approx 5.3531. \]

This yields the specific solution \( T = 70 + 305 e^{-5.3531t} \).

To find the time at which the stone reaches 90°F, we set \( T = 90 \) and solve for \( t \):

\[ 90 = 70 + 305 e^{-5.3531t} \quad \Rightarrow \quad t = -\frac{1}{5.3531} \ln\left(\frac{20}{305}\right) \approx 0.50897. \]

Thus the pizza stone will reach 90°F in about 0.51 hours.

**Example 2.1.8** Consider a tank that contains 50 gal of a solution composed of 90% water and 10% alcohol. A second solution containing 50% alcohol is added to the tank at the rate of 2 gal/min. At the same time, solution is being drained from the tank at the rate of 3 gal/min. Assuming the tank is continuously stirred, describe the volume of alcohol in the tank, and the concentration of alcohol, at any time.

**Solution.** Let \( y(t) \) represent the volume (in gal) of alcohol in the tank at time \( t \) (in min). In Section 1.3 we derived the following general model for \( y \):

\[ \frac{dy}{dt} = \left( \text{Rate of sol flowing in} \right) \cdot \left( \text{Conc of sol flowing in} \right) - \left( \text{Rate of sol flowing out} \right) \cdot \left( \frac{y}{\text{Vol of sol in tank}} \right). \]

Since the rate of solution flowing in is different than the rate flowing out, the volume of solution in the tank is not constant. To describe this volume, we can use the rate of
2.1 Linear first-order differential equations

accumulation principle:

\[ \text{Change in vol of sol in tank} = 2 \frac{\text{gal}}{\text{min}} - 3 \frac{\text{gal}}{\text{min}} = -1 \frac{\text{gal}}{\text{min}}. \]

This means that the overall volume of solution in the tank is decreasing by 1 gal/min so that the volume of solution in the tank at time \( t \) is \((50 - t) \text{ gal} \). Therefore, the model for \( y \) is

\[ \frac{dy}{dt} = 2(0.5) - 3 \left( \frac{y}{50 - t} \right) = 1 - \frac{3y}{50 - t}. \]

We are told that initially the solution in the tank is 90% alcohol. Since the tank originally contained 50 gal, the initial amount of alcohol in the tank is \( 50(0.9) = 45 \). This means we have the initial condition \( y(0) = 45 \). This is a linear first-order IVP which we can solve using the integrating factor method.

Step 1: In standard form, the DE is \( y' + \frac{3}{50 - t} y = 1 \).

Step 2: \( \mu = e^{\int \frac{3}{50 - t} dt} = e^{-3\ln(50 - t)} = (50 - t)^{-3} \)

Step 3: \( y'(50 - t)^{-3} + \frac{3}{50 - t} y(50 - t)^{-3} = 1(50 - t)^{-3} \) \( \Rightarrow \) \( \frac{d}{dt} \left( y(50 - t)^{-3} \right) = (50 - t)^{-3} \)

Step 4: \( \int \frac{d}{dt} \left( y(50 - t)^{-3} \right) dt = \int (50 - t)^{-3} dt \) \( \Rightarrow \) \( y(50 - t)^{-3} = \frac{1}{2}(50 - t)^{-2} + C \)

Step 5: \( y = \frac{1}{2}(50 - t) + C(50 - t)^{3} \)

The initial condition \( y(0) = 45 \) yields

\[ 45 = y(0) = \frac{1}{2}(50 - 0) + C(50 - 0)^{3} = 25 + 50^{3}C \] \( \Rightarrow \) \( C = 0.00016 \)

which gives the specific solution

\[ y(t) = 0.00016(50 - t)^{3} - \frac{1}{2} t + 25 = 0.00016(50 - t)^{3} + \frac{50 - t}{2}. \]

This function describes the volume of alcohol in the tank at any point in time. Note that the tank is completely drained at time \( t = 50 \) so this function is defined only for \( 0 \leq t \leq 50 \).

The concentration of the solution in the tank at time \( t \), \( c(t) \), is then

\[ c(t) = \frac{y}{\text{Vol of sol in tank}} = \frac{y}{50 - t} = 0.00016(50 - t)^{2} + \frac{1}{2}. \]
Graphs of \( y(t) \) and \( c(t) \) are shown in Figure 2.3. Note that as \( t \to 50 \), \( y(t) \to 0 \) and \( c(t) \to 0.5 \). This should be expected because the total volume of solution in the tank is approaching 0, so the volume of alcohol in the tank is also approaching 0. Also, the remaining solution in the tank is being replaced by the 50\% solution being added.

![Graphs of y(t) and c(t)](image)

**Figure 2.3:** Volume and Concentration of Alcohol

**Example 2.1.9** For a series circuit with one resistor and one inductor, called an RL circuit (see Section 1.4), we have the linear first-order differential equation \( L \frac{dI}{dt} + RI = V(t) \) where \( L \) is the inductance of the inductor, \( R \) is the resistance of the resistor, \( V(t) \) is the voltage source (the input) and \( I \) is the current. Assume \( R = 1 \) ohm and \( L = 1 \) henry. Also assume an initial condition of \( I(0) = 0 \) (no current flowing).

Find the voltage across the resistor (the output) of this circuit for the inputs \( V(t) = 12 \) volts (a DC circuit) and for \( V(t) = 120 \cos(t) \) volts (an AC circuit). For both cases graph the input and output voltages on one graph, and describe the long-term behavior of the output.

**Solution.** We want to solve \( \frac{dI}{dt} + I = V(t) \) for \( V(t) = 12 \) and for \( V(t) = 120 \cos(t) \). For either case the integrating factor is \( \mu = e^{\int 1 \, dt} = e^t \). Multiplying through by \( \mu \) and rewriting the left side as the derivative of a product we get \( \frac{d}{dt}(ye^t) = V(t)e^t \). Integrating both sides and solving for \( y \) results in the general solution \( y = e^{-t} \int V(t)e^t \, dt + ce^{-t} \).

For the case \( V(t) = 12 \) we get \( y = e^{-t} \int 12e^t \, dt + ce^{-t} = 12 + ce^{-t} \). Applying the initial condition \( y(0) = 0 \) we get \( 12 + c = 0 \) so \( c = -12 \). The particular solution is \( y = 12 - 12e^{-t} \).
For the case $V(t) = 120 \cos(t)$ we get $y = e^{-t} \int 120 \cos(t) e^t dt + ce^{-t} = 60 \cos t + 60 \sin t + \frac{1}{6} 120 \cos t$. Using $y(0) = 0$ results in $60 + c = 0$ so $c = -60$. Thus $y = 60 \cos t + 60 \sin t - 60e^{-t}$.

We graph the results for both cases in Figure 2.1.9. For the DC circuit, the voltage across the resistor increases from zero and asymptotically approaches the input voltage of 12. For the AC circuit, the voltage across the resistor approaches a steady oscillation, with a smaller amplitude than the input, and which lags slightly behind the input (the peaks of the output occur a bit behind those of the input).

![Figure 2.4: Input and output for (a) DC circuit and (b) AC circuit](image)

**Exercises**

2.1.1 Find a general solution to the basic population model $y' = ky$.

For each equation below state whether it is linear or not.

1. $x' = t \sin(x)$
2. $x' = t + \sin(x)$
3. $x' = x \sin(t)$
4. $x' = x + \sin(t)$
5. $y' + ty^2 = t$
6. $y' + x^2y = x^2$

Solve each of the following linear equations.

7. $x' + 2x = e^{-2t} \cos(t)$
8. $x' = x + 1$
9. $tx' + 2x = 1 + t$
10. $x' = x + t$

Solve the following initial-value problems:

11. $x' = x + e^{3t}, \ x(0) = 2$
12. $x' + 2x = e^{-2t} \cos(t), \ x(0) = 1$
13. $tx' + x = t^2 + t, \ x(1) = 0$
14. $x' + 2tx = t, \ x(0) = 1$

2.2 Existence, uniqueness, and portraits for first-order equations

Up to now, we have talked about how to describe the solution(s) to a differential equation in different ways: explicit, numerical, and graphical. In this section we discuss three somewhat more theoretical questions about IVP’s:

1. When do we know that a solution to an initial value problem exists?
2. If a solution does exist, is it the only one?
3. If a solution does exist, on what interval of the independent variable is it valid?

When we talk about existence in this section, we do not mean existence of an explicit solution in terms of elementary functions. Any function will do, whether given in terms of infinite series, integrals, or other limiting processes. In other words, any way of generating a value of the dependent variable for a given value of the independent variable is acceptable as long as the resulting function “works” as a solution to the differential equation.

We start with a theorem that answers all three of these questions for linear first-order IVP’s. In the proof we derive a formula for the specific solution to such an IVP. This formula is based on the formula for a general solution to a first-order DE $y' + g(x)y = h(x)$ found in step 5 of the integrating factor method,

$$y = \mu^{-1} \int h(x) dx + C \mu^{-1}$$

where $\mu = e^{\int g(x) dx}$.
2.2 Existence, uniqueness, and portraits for first-order equations

Theorem 2.1 (Existence/Uniqueness for Linear First-Order IVP’s) Consider an IVP of the form

\[ y' + g(x) y = h(x), \quad y(x_0) = y_0. \]

Assume that \( g(x) \) and \( h(x) \) are both continuous on some interval \( a < x < b \) and that \( a < x_0 < b \). Then there exists a unique solution \( y(x) \) to the initial value problem that is defined on \( a < x < b \).

Proof. To prove existence, consider the function

\[ y(x) = \frac{1}{\mu(x)} \int_{x_0}^{x} h(s)\mu(s) \, ds + y_0 \frac{1}{\mu(x)} \]

where \( \mu(x) = e^{\int_{x_0}^{x} g(s) \, ds} \). We will show that \( y(x) \) is a solution to the IVP. First note that

\[ \mu(x_0) = e^{\int_{x_0}^{x_0} g(s) \, ds} = e^0 = 1. \]

Thus

\[ y(x_0) = \frac{1}{\mu(x_0)} \int_{x_0}^{x_0} h(s)\mu(s) \, ds + y_0 \frac{1}{\mu(x_0)} = \frac{1}{1} \cdot 0 + y_0 \cdot \frac{1}{1} = y_0 \]

showing that this function satisfies the initial condition. To show that this function satisfies the DE, note that by the Fundamental Theorem of Calculus,

\[ \frac{d}{dx} \mu(x) = e^{\int_{x_0}^{x} g(s) ds} \cdot \frac{d}{dx} \int_{x_0}^{x} g(s) ds = \mu(x) g(x). \]

For notational simplicity, let \( a(x) = \int_{x_0}^{x} h(s)\mu(s) \, ds \) so that \( y(x) = \frac{1}{\mu} \cdot a + y_0 \frac{1}{\mu} \) (we have further simplified notation by dropping the \( x \)). Then \( \frac{d}{dx} a(x) = h(x)\mu(x) \) and

\[ \frac{d}{dx} y(x) = -\frac{1}{\mu^2} \cdot \mu' \cdot a + \frac{1}{\mu} \cdot a' - y_0 \frac{1}{\mu^2} \cdot \mu' \]

\[ = -\frac{1}{\mu^2} \cdot \mu \cdot g \cdot a + \frac{1}{\mu} \cdot h \cdot \mu - y_0 \frac{1}{\mu^2} \cdot \mu \cdot g \]

\[ = -\frac{1}{\mu} \cdot g \cdot a + h - y_0 \frac{1}{\mu} \cdot g. \]

Then

\[ y' + g \cdot y = -\frac{1}{\mu} \cdot g \cdot a + h - y_0 \frac{1}{\mu} \cdot g + g \cdot \frac{1}{\mu} \cdot a + g \cdot y_0 \frac{1}{\mu} = h \]

Thus \( y(x) \) solves the DE which proves that a solution to the IVP does exist. Note that we
may not be able to evaluate this formula for \( y(x) \) in terms of elementary functions, but this was never claimed in the theorem.

Note also that this solution is defined on \( a < x < b \), because the integrals

\[
\int_{x_0}^{x} g(s) \, ds \quad \text{and} \quad \int_{x_0}^{x} h(s)\mu(s) \, ds
\]

which appear in the formula for \( y(x) \) both exist on \( a < x < b \) due to the continuity of \( g(x) \) and \( h(x) \) on the interval \( a < x < b \).

To prove uniqueness, suppose that \( y_1(x) \) and \( y_2(x) \) both satisfy the IVP. Define \( z(x) = y_1(x) - y_2(x) \). Then

\[
z' + g \cdot z = y_1' - y_2' + g \cdot y_1 - g \cdot y_2 = (y_1' + g \cdot y_1) + (y_2' + g \cdot y_2)
\]

Since both \( y_1 \) and \( y_2 \) satisfy the DE,

\[(y_1' + g \cdot y_1) + (y_2' + g \cdot y_2) = h - h = 0\]

and hence

\[z' + g \cdot z = 0.\]

Solving this DE with the integrating factor method, we get \( z = Ce^{-\int g \, dx} \) and that

\[z(x_0) = y_1(x_0) - y_2(x_0) = y_0 - y_0 = 0.\]

But the only way that an exponential function such as \( Ce^{-\int g \, dx} \) can equal 0 for even one value of \( x \) is if \( C = 0 \). This means that \( z(x) = 0 \) for all \( x \) and hence

\[y_1(x) = y_2(x)\]

for all \( x \), verifying uniqueness.

A solution to an IVP as described in Theorem 2.1 is a function. It may appear that this function could be evaluated at many different values of \( x \). But the theorem states that the solution only exists on an interval on which \( g(x) \) and \( h(x) \) are both continuous and \( x_0 \) must belong to this interval. The next example illustrates this point.

**Example 2.2.1** Solve the initial value problem \( xy' + y = 10 \), \( y(1) = 1 \), and determine the largest interval of existence.

**Solution.** We use the integrating factor method to find a general solution:
Step 1: \( y' + \frac{y}{x} = \frac{10}{x} \)

Step 2: \( \mu = e^{\int \frac{1}{x} \, dx} = e^{\ln x} = x \)

Step 3: \( xy' + x \cdot \frac{y}{x} = x \cdot \frac{10}{x} \Rightarrow \frac{d}{dx}(xy) = 10 \)

Step 4: \( \int d(xy) = \int 10 \, dx \Rightarrow xy = 10 + C \)

Step 5: \( y = 10 + \frac{C}{x} \)

Using \( y(1) = 1 \) we get \( 1 = 10 + C \) so \( C = -9 \). Thus the particular solution to the IVP is \( y = 10 - \frac{9}{x} \).

Note that this solution is a function. It appears that we could evaluate this function at any \( x \neq 0 \). However, we have \( g(x) = \frac{1}{x} \) and \( h(x) = \frac{10}{x} \) which are both continuous on the intervals \( x < 0 \) and \( x > 0 \). Since \( x_0 = 1 \) is part of the interval \( x > 0 \), the largest interval of existence is also \( x > 0 \). This means that the solution \( y = 10 - \frac{9}{x} \) is defined only for \( x > 0 \). \( \square \)

Theorem 2.1 applies to first-order IVP's of a certain form. We now state an existence and uniqueness theorem for any first-order IVP. This theorem applies to both linear and nonlinear problems.

**Theorem 2.2 (Existence/Uniqueness for General First-Order IVP’s)** Consider a first-order IVP of the form

\[ y' = f(x, y), \quad y(x_0) = y_0. \]

Assume that \( f(x, y) \) and \( \frac{\partial f}{\partial y}(x, y) \) are both continuous on a region in the \( xy \)-plane given by \( a < x < b \) and \( c < y < d \), and that this region contains the point \( (x_0, y_0) \). Then there exists a unique solution \( y(x) \) to the IVP, defined on some interval \( \alpha < x < \beta \) which is contained in the interval \( a < x < b \).

**Proof.** The proof of this theorem is beyond the scope of this text. The reader is referred to the classic text *Theory of Ordinary Differential Equations* by Earl Coddington and Norman Levinson (1955), or to the website [http://en.wikipedia.org/wiki/Picard-Lindelof_theorem](http://en.wikipedia.org/wiki/Picard-Lindelof_theorem).

Note that with this theorem, the region \( a < x < b, c < y < d \) where the premises of the theorem are satisfied can be large, and yet the interval \( \alpha < x < \beta \) can be quite small. One important difference between Theorems 2.1 and 2.2 is that for linear differential equations,
the largest interval of existence can be determined before the solution is found, whereas for nonlinear equations, generally one must first find the solution and then determine the interval of existence from that.

In the next section we present a method for solving certain types of nonlinear DE’s (ones that are called separable). For the next example, we resort to computer algebra to determine a solution to a nonlinear equation, and then from it we state the interval of existence.

**Example 2.2.2** Use available software to find a solution to the IVP $y' = y^2$, $y(0) = 5$, and then determine the largest interval on which that solution is defined. Is your solution the only one for which $y(0) = 5$?

**Solution.** First note that $f(x, y) = y^2$ and $\frac{\partial F}{\partial y} = 2y$, both of which are continuous everywhere in the $xy$-plane. This region certainly contains the point $(0, 5)$. Thus Theorem 2.2 applies to this IVP.

The solution given by Wolfram Alpha is

$$y = \frac{5}{1 - 5x}.$$

We could evaluate this function for $x < \frac{1}{5}$ and $x > \frac{1}{5}$. However, since $x_0 = 0 < \frac{1}{5}$, we conclude that as a solution to the IVP, this function is defined for only $x < \frac{1}{5}$. Theorem 2.2 guarantees that this solution is unique.

One thing to take away from the previous example is that solutions to differential equations can’t “jump over” asymptotes. The function $y = \frac{5}{1 - 5x}$ has a vertical asymptote at $x = \frac{1}{5}$. However, the solution to the IVP is only defined for $x < \frac{1}{5}$. This means that we must be careful when using explicit solutions to nonlinear differential equations. Suppose that we wanted to know the value of $y(1)$. It is tempting, but quite wrong, to just substitute $x = 1$ into the explicit solution $y = \frac{5}{1 - 5x}$ to get $y = -\frac{5}{4}$. The absurdity of this answer becomes obvious when we look at a slope field in Figure 2.5.

The slope fields shows that all solution curves must be increasing from left to right, since all slope marks have positive or zero slopes. Therefore, the solution curve depicted cannot go through the point $(1, -5/4)$ because the curve would have to decrease.

In previous examples we have seen solutions to IVP’s $y(x)$ such that $\lim_{x \to \infty} y(x) = \infty$. Note that in Example 2.2.2, we have $\lim_{x \to 1/5^-} y(x) = \infty$. The fact that this limit is infinite for $x$ approaching a finite number is referred to as finite time blow-up. In such a case, it makes no sense to talk about the long-term behavior (as $x \to \infty$) of the solution, since the solution “ends” at a finite value of the independent variable. Finite time blow-up can occur for either linear or nonlinear DE’s.
Portraits of first-order equations

We have seen one or more solution curves graphed on the same plane. A portrait of a first-order differential equation is a plot of many solution curves, so as to “fill out” the plot region. A portrait provides similar information as to that of a slope field, in that if enough solution curves are plotted one should be able to determine the path of any solution curve given an initial condition. Often a portrait and a slope field are plotted together.\footnote{The term portrait is not standard in the differential equations literature. We use it here for first-order differential equations in order to have a term parallel to phase portrait (which is standard usage) for a plot of one dependent variable against the other for a system of two first-order autonomous equations (to be discussed in Chapters 3 and 4).}

Example 2.2.3 Create a portrait along with slope field of the differential equation $y' = y^2 - t$. Use the plot region $-1 \leq t \leq 5$, $-3 \leq x \leq 3$ Use the Example 2.2.3 applet at uhaweb.hartford.edu/rdecker/DeckerDEbook/DeckerDEbook.html or other available software.

Solution. Many initial conditions must be used to get a good portrait. Some software require you to type in a large number of initial conditions. Some, such as the applet referenced above, provide an interactive tool to choose initial conditions. To get a good portrait, you should use initial conditions spread throughout the graph window (not just, say, along the $y$-axis). We used the applet to obtain Figure 2.6.

The uniqueness part of Theorems 2.1 and 2.2 is intimately tied up with the interpretation
CHAPTER 2  First-order Differential Equations

Figure 2.6: Portrait and slope field of \( y' = y^2 - t \)

of portraits of differential equations. We state the following theorem, then discuss its connection with differential equation portraits.

Theorem 2.3 (No intersections theorem) Consider a first-order DE of the form

\[ y' = f(x, y). \]

Let \( I \) be a region in the \( xy \)-plane where \( f(x, y) \) and \( \frac{\partial f}{\partial y}(x, y) \) are continuous. Then no two distinct solution curves to \( y' = f(x, y) \) can intersect anywhere in \( I \).

Proof. Let \( (x_0, y_0) \) be an arbitrary point in region \( I \). This point corresponds to the initial condition \( y(x_0) = y_0 \). The premises of Theorem 2.2 are satisfied on \( I \), so by this theorem there is a unique solution satisfying this initial condition. Graphically, this means that there is exactly one solution curve through this point. In other words, there cannot be two distinct curves through this point.

If we have a region where solution curves cannot intersect, then we can use a “squeeze theorem” type argument to approximate a new solution curve given two other curves as illustrated in the next example.

Example 2.2.4 Use the portrait in Figure 2.6 to estimate \( y(3) \) given that \( y(1) = 0 \) for the differential equation \( y' = y^2 - t \).

Solution. We have \( f(t, y) = y^2 - t \) and \( \frac{\partial f}{\partial y}(t, y) = 2y \). Since both are continuous everywhere, the solution curves cannot cross anywhere in the \( ty \)-plane. Also note that this IVP will have a unique solution. Let \( y(t) \) denote this solution.
Figure 2.6 is reproduced in Figure 2.7. The solution curve \( y(t) \) will go through the point \((1, 0)\). Notice that this point is “between” the two highlighted solution curve. Since solution curves cannot cross, the curve \( y(t) \) must stay between these two highlighted curves.

![Figure 2.7: Portrait and slope field of the equation \( y' = y^2 - t \)](image)

Moving left to right, the two highlighted curves get closer to one another and are nearly indistinguishable at \( t = 3 \). The corresponding \( y \)-value is approximately \(-1.6\). Thus the two highlighted curves “squeeze” the solution curve \( y(t) \) through the point \((3, -1.6)\) and we conclude that \( y(3) \approx -1.6 \).

Theorem 2.3 says that on regions where \( f(x, y) \) and \( \frac{\partial f}{\partial y}(x, y) \) are continuous, solution curves cannot cross. On regions where \( f(x, y) \) and \( \frac{\partial f}{\partial y}(x, y) \) are not continuous, it is possible that solution curves can cross. The next example illustrates this.

**Example 2.2.5** For each DE below, determine any regions in the \( xy \)-plane where solution curves could intersect. If solution curves can intersect, find at least two distinct solution curves that do intersect.

a. \( y' = \frac{y}{x} \)  
   b. \( y' = y + x \)  
   c. \( y' = \sqrt{y} \)

**Solution.**

a. We have \( f(x, y) = \frac{y}{x} \) and \( \frac{\partial f}{\partial y} = \frac{1}{y} \). The only points of discontinuity of either function are along the line \( x = 0 \), therefore the \( y \)-axis is the only region where solution curves could intersect.

   One possible initial condition on the line \( x = 0 \) is \( y(0) = 0 \). The initial value problem \( y' = \frac{x}{y}, y(0) = 0 \) can be solved using a technique called separation of variables (which
we cover in the next section), or we can use software. The general solution is \( y = Cx \).

Since the general solution satisfies \( y(0) = 0 \) for any value of \( C \), there are infinitely many solutions to the IVP, all of which intersect at the point \((0, 0)\).

b. We have \( f(x, y) = y + x \) and \( \frac{\partial f}{\partial x}(x, y) = 1 \). These functions are both continuous for all \( x \) and \( y \), therefore solution curves cannot intersect anywhere.

c. We have \( f(x, y) = \sqrt{y} = y^{1/2} \) so that \( \frac{\partial f}{\partial y}(x, y) = \frac{1}{2\sqrt{y}} \). Note that \( f(x, y) \) is defined and continuous for \( y \geq 0 \) and \( \frac{\partial f}{\partial y}(x, y) \) is discontinuous (and undefined) along \( y = 0 \). Therefore solution curves could intersect along \( y = 0 \).

If we choose the initial condition \( y(0) = 0 \), then the specific solution is \( y(t) = \frac{t^2}{4} \) for \( x \geq 0 \). Another solution to the same initial value problem is \( y(x) = 0 \) for all \( x \) (see Exercise 2.2.2 for a verification of these solutions). These two solutions intersect at the point \((0, 0)\).

The last example illustrates that we must be very careful in using the predictions of any mathematical model, especially a DE which may not have a unique solution.

**Example 2.2.6** (The miraculous population model) Consider the basic population model \( y' = ky \). If the initial condition is \( y(0) = 0 \), then a solution is \( y(t) = 0 \) for all \( t \). By Theorem 2.1, this solution is unique. This should should make intuitive sense. If we start with zero population, we should always have zero population.

What if the population model were \( y' = k\sqrt{y} \)? This DE models the assumption that the rate of growth of the population is proportional to the square root of the size of the population. Note that as \( y \) increases, \( \sqrt{y} \) also increases, which means that \( y' \) would also increase. This does not seem too unreasonable. If \( k = 1 \) and we have the initial condition \( y(0) = 0 \), we get two possible solutions for \( t \geq 0 \):

\[
y(t) = \frac{t^2}{4} \quad \text{and} \quad y(t) = 0.
\]

The solution \( y(t) = 0 \) makes perfect sense. However, the solution \( y(t) = \frac{t^2}{4} \) predicts that the population would increase for \( t > 0 \) even though we started with a population of 0. It does not seem reasonable that this could actually happen (spontaneous generation of living organisms was once considered possible, but an experiment by Louis Pasteur in 1859 is considered to have ruled it out).

The solution \( y(t) = \frac{t^2}{4} \) is much more interesting than \( y(t) = 0 \). However, this example shows that the more interesting solution is not necessarily correct.
Exercises

2.2.1 Use the integrating factor method to show that a general solution to the DE $z' + gz = 0$ is $z = Ce^{-\int g \, dx}$.

2.2.2 Consider the IVP

$$y' = \sqrt{y}, \quad y(0) = 0.$$ 

a. Verify that $y = 0$ is a specific solution to this IVP.

b. Verify that $y = x^2/4$ is a specific solution to this IVP.

c. Explain why the solution $y = x^2/4$ is defined for only $x \geq 0$. (Hint: Plug the solution into the DE. What is the sign of the right-hand side? What is the sign of the left-hand side if $x < 0$?)

2.2.3 Show that the differential equation $x' = x \frac{x}{1+x^2}$ has a unique solution through every initial point $(t_0, x_0)$. Can solution curves ever intersect for this differential equation?

2.2.4 Does the equation $x' = x^2 - t$ have a unique solution through every initial point $(t_0, x_0)$? Can solution curves ever intersect for this differential equation? If so, where?

2.2.5 Does the equation $x' = x^{2/3}$ have a unique solution through every initial point $(t_0, x_0)$? Can solution curves ever intersect for this differential equation? If so where? Can you find two solution curves that cross each other?

Use available software to create a detailed slope field and phase portrait for each equation below. Briefly describe the long-term behavior for various values of the initial condition $x(0)$.

2.2.6 $x' = x + t/2$.

2.2.7 $x' = x^2(1 - x)$

2.2.8 $x' = 1 + t/2$

2.2.9 $\frac{dP}{dt} = rP(1 - \frac{P}{N})$ using the parameter values $r = 0.5$ and $N = 10$

2.3 Separable Differential Equations

Like a linear equation, a separable DE is an equation that can be written in a special form which leads to a solution method. We begin with a definition.
Definition 2.3.1 A first-order differential equation \( y' = f(x, y) \) is said to be separable if \( f(x, y) \) can be factored into the form

\[
f(x, y) = g(x)h(y)
\]

where \( g(x) \) is a function of only \( x \) and \( h(y) \) is a function of only \( y \).

As with the integrating factor method for linear equations, the names of the dependent and independent variables are not important. For example if we wrote the DE as \( x' = f(t, x) \), with \( t \) the independent variable and \( x \) the dependent variable, the requirement for separability would be \( f(t, x) = g(t)h(x) \).

Example 2.3.1 Determine which of the following equations are separable.

a. \( y' = xy \)  
b. \( x' = xt + x \)  
c. \( y' = x + y \)  
d. \( y' = 3e^{t+y} \)  
e. \( x' = \frac{\sin t}{\cos x} \)  
f. \( y' = y \)

Solution.

a. Clearly separable with \( g(x) = x \) and \( h(y) = y \).

b. Separable, since \( xt + x \) can be factored into \( x(t + 1) \). Then \( f(t) = t + 1 \) and \( g(x) = x \).

c. The right-hand side, \( x + y \), cannot be factored into the form \( g(x)h(y) \) so this DE is not separable. While we could factor as \( x(1 + y/x) \), this is no help as \( (1 + y/x) \) is not a function of \( y \) alone.

d. Separable with \( 3e^{t+y} = 3e^t e^y \) so \( g(t) = 3e^t \) and \( h(y) = e^y \).

e. Separable since

\[
\frac{\sin t}{\cos x} = \sin t \left( \frac{1}{\cos x} \right).
\]

f. Since the independent variable does not appear in the equation (such equations are called autonomous and will be studied in a later section) we can choose the name of the independent variable; let’s choose \( x \). This equation is separable since \( y = 1 \cdot y \) so that \( g(x) = 1 \) and \( h(y) = y \). Note that \( g(x) = 1 \) can be considered to be a constant function of \( x \). Autonomous first-order equations are always separable.
The next example helps motivate a relatively simple technique for finding an explicit solution to a separable DE.

**Example 2.3.2** Consider the separable DE \( y' = xy \). For simplicity we will add the restriction that \( y > 0 \). We can algebraically rewrite this DE as

\[
\frac{y'}{y} = x .
\]  \( (2.2) \)

On the left-hand side of equation (2.2), we have the derivative of a function divided by the function. This should remind us of the general logarithmic rule from Calculus:

\[
\frac{d}{dx} \ln [u(x)] = \frac{u'}{u}
\]

where \( u(x) > 0 \). So we can rewrite equation (2.2)

\[
\frac{d}{dx} \ln y = x .
\]  \( (2.3) \)

Our goal is to get a formula for \( y \). We are getting closer because we have gotten rid of \( y' \). But now we have introduced a \( \frac{d}{dx} \). To get rid of \( \frac{d}{dx} \), let’s try writing the right-hand side of the DE in terms of a derivative with respect to \( x \). Note that

\[
\frac{d}{dx} \left( \frac{x^2}{2} \right) = x .
\]  \( (2.4) \)

Thus we can rewrite equation (2.3) as

\[
\frac{d}{dx} \ln y = \frac{d}{dx} \left( \frac{x^2}{2} \right) .
\]

Now, \( \ln y \) and \( x^2/2 \) are functions whose derivatives are equal for all \( x \). The only way this is possible is if these two functions differ by a constant. Thus we have

\[
\ln y = \frac{x^2}{2} + C
\]  \( (2.5) \)

where \( C \) is an arbitrary constant. Now compare equations (2.2) and (2.5). Note that the left-hand side of (2.5) is simply the antiderivative of the left-hand side of (2.2) where the variable of integration is \( y \). The right-hand sides share a similar relationship where the
variable of integration is $x$. In mathematical notation, we have shown that
\[
\int \frac{y'}{y} \, dy = \int x \, dx.
\]
To finish the solution, we solve equation (2.5) for $y$:
\[
e^{\ln y} = e^{x^2/2+C} \quad \Rightarrow \quad y = e^{x^2/2+C}.
\]
This gives us an explicit general solution to the original DE.

We generalize this example in the method of separation of variables.

---

**The Method of Separation of Variables**

**Purpose**: To solve a DE of the form $y' = g(x)h(y)$.

**Step 1**: Replace $y'$ with $\frac{dy}{dx}$ and rewrite the DE in the form
\[
\frac{1}{h(y)} \, dy = g(x) \, dx.
\]

**Step 2**: Add an integral sign to both sides to get
\[
\int \frac{1}{h(y)} \, dy = \int g(x) \, dx
\]
and then integrate both sides (if possible). Make sure to add a constant to the result on the right-hand side.

**Step 3**: Solve the equation that results from step 2 for the dependent variable (if possible).

---

**Example 2.3.3** Use separation of variables to find the general solution to $y' = xy^2$.

**Solution.** We follow the three steps:

**Step 1**: \[ \frac{dy}{dx} = xy^2 \quad \Rightarrow \quad \frac{1}{y^2} \, dy = x \, dx \]
Step 2: \( \int \frac{1}{y^2} \, dy = \int x \, dx \Rightarrow \quad \frac{1}{y} = \frac{1}{2}x^2 + c \text{ where } c \text{ is an arbitrary constant} \)

Step 3: \( y = \frac{1}{\frac{1}{2}x^2 + c} \)

The expression for \( y \) in step 3 is OK, but we usually don’t like fractions within fractions. So we can do a little bit of algebra to the right-hand side to simplify things:

\[
y = \frac{1}{\frac{1}{2}x^2 + c} \cdot \frac{2}{2} = \frac{2}{x^2 + 2c}.
\]

To further simplify things, note that \( c \) is an arbitrary constant, so \( 2c \) is an arbitrary constant. Thus we define the arbitrary constant \( C = 2c \) and write the final solution as

\[
y = \frac{2}{x^2 + C}.
\]

We always want to write the final solution in as simple terms as possible.

**Example 2.3.4** Use separation of variables to find the general solution to \( y' = y \).

**Solution.** Note that the independent variable is not explicitly given. For something different, we’ll use \( t \).

\[
\begin{align*}
\text{Step 1:} & \quad \frac{dy}{dt} = y \quad \Rightarrow \quad \frac{1}{y} \, dy = 1 \, dt \\
\text{Step 2:} & \quad \int \frac{1}{y} \, dy = \int 1 \, dt \quad \Rightarrow \quad \ln |y| = t + c \\
\text{Step 3:} & \quad e^{\ln |y|} = e^{t+c} \quad \Rightarrow \quad |y| = e^{t+c}
\end{align*}
\]

Note that we put the absolute value signs around \( y \) in step 2 because we are not told that \( y \) is always positive. As in the previous example, this description for \( y \) is OK, but we can simplify it. Note that by properties of exponents we can write

\[
|y| = e^{t+c} = e^t e^c.
\]

But \( c \) is a constant, so \( e^c \) is a constant. However, note that \( e^c \) is positive. So define \( C \) to be a positive constant. Then the solution can be written as

\[
|y| = Ce^t.
\]
To further simplify this solution, note that if \( y \geq 0 \), then
\[
|y| = y \quad \Rightarrow \quad y = Ce^t
\]
and if \( y < 0 \), then
\[
|y| = -y \quad \Rightarrow \quad y = -Ce^t.
\]
We can combine these two cases and allow for the possibility that \( y \) is positive or negative by allowing \( C \) to be a positive or negative constant and writing the general solution as
\[
y = Ce^t.
\]
Changing the constant as we have done in the last two examples is sometimes referred to as “abusing the constant.”

Separation of variables can fail to yield an explicit solution if the integrals in step 2 cannot be evaluated in terms of elementary functions, or we cannot solve for \( y \) in step 3. The next two examples illustrate such cases.

**Example 2.3.5** Attempt to use separation of variables to find the general solution to \( x' = xe^{xt} \).

**Solution.** Note that the dependent variable is \( x \) and the independent variable is \( t \).

Step 1: \( \frac{dx}{dt} = xe^{xt} \quad \Rightarrow \quad \frac{1}{xe^x} \, dx = t \, dt \)

Step 2: \( \int \frac{1}{xe^x} \, dx = \int t \, dt \)

The integral on the left cannot be expressed in terms of elementary functions. At the time of the writing of this text, Wolfram Alpha gives \( Ei(-x) \), Maple returns \(-Ei(1, x)\), and the TI89 returns \( \int \frac{e^x}{x} \, dx \) (which means the TI89 cannot find the integral). When a special function such as \( Ei(-x) \) or \(-Ei(1, x)\) appears in the result, one can consult the help files for the particular system used in order to find the definition of the special function. For our purposes, we can say that the method of separation of variables fails to give us an explicit solution in terms of elementary functions.

**Example 2.3.6** Use separation of variables to find the general solution to \( x' = \frac{t}{xe^x} \).

**Solution.** We follow the 3 steps:

Step 1: \( \frac{dx}{dt} = \frac{t}{xe^x} \quad \Rightarrow \quad xe^x \, dx = t \, dt \)
Step 2: \[ \int xe^x \, dx = \int t \, dt \implies xe^x - e^x = \frac{1}{2} t^2 + C. \] The integral on the left is evaluated using integration by parts or software.

Step 3: We try to use software to solve for \( x \). Wolfram Alpha runs out of time, Maple returns \( \text{RootOf} \left( \exp(\_Z) \_Z - \_Z - t - c \right) \), and the TI89 returns \( \text{solve}(2xe^x - 2x - t^2 - 2c = 0) \).

Since none of the software returns an explicit result for \( x \) in terms of \( t \), we conclude that no explicit solution for \( x \) in terms of elementary functions of \( t \) is possible. This is often the case when the variable that we are solving for (\( x \) in this case) appears both in the argument of an exponential function and outside of the same exponential function.

Applications

Several of the models we have seen so far in this text are separable. We investigate a few of these below, and more in the exercises.

**Example 2.3.7** Consider the basic population model \( y' = ky \). Suppose that \( y \) represents the size of a bacteria population in a patient in the early stages of an infection and that there are 1000 bacteria present at a certain point in time \( (t, \text{in hours}) \) which we mark with the value \( t = 0 \). If there are 3000 present two hours later, how many will be present after five hours? Support your answer graphically.

**Solution.** The DE \( y' = ky \) is separable, so we apply the method separation of variables to find a general solution.

Step 1: \[ \frac{dy}{dt} = ky \implies \frac{1}{y} \, dy = k \, dt \]

Step 2: \[ \int \frac{1}{y} \, dy = \int k \, dt \implies \ln y = kt + c \]

Step 3: \[ e^{\ln y} = e^{kt+c} \implies y = e^{t+c} = e^{kt}e^{c} = Ce^{kt} \text{ where } C > 0 \text{ is a constant} \]

Note that we don’t need to use \( \ln |y| \) for the antiderivative of \( 1/y \) in step 2 since we know that \( y \) must be positive. In Example 1.1.8 we verified that a function of the form \( y = Ce^{kt} \) is a general solution to the DE. The calculations above show where the solution came from. Now we summarize the information given in the problem in the form of a table so we can determine the constants \( C \) and \( k \).

<table>
<thead>
<tr>
<th>( t )</th>
<th>0</th>
<th>2</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>1000</td>
<td>3000</td>
<td>?</td>
</tr>
</tbody>
</table>
Using the first data point we get \(1000 = Ce^{k(0)} = C\) and the second gives
\[
3000 = 1000e^{k(2)} \implies k = \frac{1}{2} \ln(3) \approx 0.54931.
\]
Thus our model is \(y = 1000e^{0.54931t}\). After five hours the population size is
\[
y(5) = 1000e^{0.54931(5)} \approx 15,589.
\]

Figure 2.8 shows a slope field for the DE \(y' = 0.549y\) with the solution curve corresponding to \(y(0) = 1000\). Notice that this curve goes through the point \((5, 15589)\). This supports our answer.

The differential equation in the previous example was both linear and separable, so it could also have been solved using the integrating factor method. The next two examples involve nonlinear equations, and so separation of variables is our only choice.

**Example 2.3.8** Consider the scenario in Example 2.3.7, with the one exception that there is a carrying capacity of 10,000 for the environment of the bacteria. Thus we use the logistic model
\[
y' = ay \left(1 - \frac{y}{10000}\right)
\]
as described in Section 1.3 instead of the basic growth model \(y' = ky\). Predict the population after 5 hours using an explicit solution and support your answer graphically.
Solution. We replace the carrying capacity with the parameter $N$ and find the general solution to the DE using separation of variables. We use software to perform the calculations.

Step 1: \( \frac{dy}{dt} = ay \left(1 - \frac{y}{N}\right) \Rightarrow \frac{1}{y\left(1 - \frac{y}{N}\right)} \, dy = a \, dt \)

Step 2: \( \int \frac{1}{y\left(1 - \frac{y}{N}\right)} \, dy = \int a \, dt \Rightarrow \ln y - \ln(y - N) = at + c \)

Step 3: \( y = \frac{N}{1 - Ce^{-at}} \) where $C > 0$ is a constant

To find $C$, consider the generic initial condition $y(0) = y_0$:

\[
y_0 = y(0) = \frac{N}{1 - Ce^{-a(0)}} \Rightarrow C = 1 - \frac{N}{y_0}.
\]

After distributing the negative sign in front of the $C$, we get the general model

\[
y = \frac{N}{1 + (N/y_0 - 1)e^{-at}}.
\]

This form is quite nice for understanding the behavior of the logistic model. It is easy to see that $y(0) = N/(N/y_0) = y_0$ and as $t \to \infty$, $y \to N$, which is just what we want to happen.

To finish our problem we substitute in $y_0 = 1000$ and $N = 10,000$ to get the model

\[
y = \frac{10000}{1 + 9e^{-at}}.
\]

The data point $y(2) = 3000$ yields

\[
3000 = \frac{10000}{1 + 9e^{-2a}} \Rightarrow a = -\frac{1}{2} \ln(7/27) \approx 0.67496.
\]

Plugging this value of $a$ into the model, we get that after 5 years there will be

\[
y = \frac{10000}{1 + 9e^{-0.67496(5)}} \approx 7645 \text{ bacteria}.
\]

Figure 2.9 provides a graphical verification of this solution.
In our next application, we want to describe the height of a liquid that is draining out of a tank through a hole in the bottom. This model is based on Torricelli’s law which is a principle in fluid dynamics that describes the speed at which the liquid flows out. In words, this law says,

The speed at which fluid flows from a hole in the bottom of a tank filled to a depth $h$ is the same as the speed an object would obtain in free fall from a height $h$.

To convert this law into a DE, we use a little physics, specifically potential and kinetic energy and the law of conservation of energy. The potential energy of an object of mass $m$ at a height $h$ above the ground is

$$E_P = mgh$$

where $g$ is the acceleration due to gravity. The kinetic energy of an object moving with velocity $v$ (speed is the absolute value of velocity) is

$$E_K = \frac{1}{2}mv^2.$$ 

The law of conservation of energy, stated in terms of a falling object, says that the potential energy lost by an object after it has fallen a certain distance equals the gain in the object’s kinetic energy. So if the object started at height $h$ and then hit the ground, its potential
energy went from \( mgh \) to 0. If the object started at rest, then its kinetic energy went from 0 to \( \frac{1}{2}mv^2 \). Note that we are ignoring any type of drag. Thus we have

\[
mgh = \frac{1}{2}mv^2.
\]

Solving this equation for \( v \) yields

\[
v = \sqrt{2gh}.
\]

This equation describes the speed at which the liquid drains out. If distance is measured in m and time in sec, then the units of \( v \) are m/sec. Let \( a \) represent the area of the hole through which the liquid drains. Its units are m\(^2\). If we multiple \( a \) and \( v \), we get

\[
a \cdot v = \left(\text{am}^2\right) \cdot \left(\sqrt{2gh} \cdot \frac{\text{m}}{\text{sec}}\right) = a\sqrt{2gh} \cdot \frac{\text{m}^3}{\text{sec}}.
\]

The units indicate that \( a \cdot v \) represents the change in volume with respect to time (the derivative of the volume). Let \( V(t) \) represent the volume of liquid in the tank at time \( t \). Since the volume is decreasing, we have \( V' = -a\sqrt{2gh} \). Now, \( a \), 2, and \( g \) are all constants, so we can rewrite this equation as

\[
V' = -a\sqrt{2gh} = -a\sqrt{2g}\sqrt{h} = -k\sqrt{h}
\]

where \( k = a\sqrt{2g} \) is a constant.\(^2\) Note that in this equation, \( h \) represents the height of water in the tank at time \( t \). Thus \( h \) is a function of \( t \). If in addition, fluid is being added at a constant rate \( r_{in} \), then we get

\[
V'(t) = r_{in} - k\sqrt{h(t)}
\]

using the rate of accumulation principle.

The problem with equation (2.7) is that it involves two dependent variables, \( V \) and \( h \). In order to solve this problem we need to relate \( V \) to \( h \), which depends on the geometry of the tank. For example, in a cylindrical tank with radius \( R \) we have

\[
V(t) = \pi R^2h \quad \Rightarrow \quad V' = \pi R^2h'
\]

which yields the DE for \( h \)

\[
\pi R^2h' = r_{in} - k\sqrt{h}. \quad (2.8)
\]

\(^2\)Determining the value of \( k \) is not as simple as measuring the area of the hole and then multiplying it by \( \sqrt{2g} \). In real-world situations, the fluid flows out in such a way that the flow-lines narrow, reducing the effective area of the hole by some contraction coefficient. This coefficient must be experimentally measured. In other words, \( k \) must be experimentally measured. See Exercise 2.3.19 for a scenario illustrating how \( k \) could be measured.
In general, if $A(h)$ is the cross-sectional area of the tank as a function of $h$ then we can write $V' = A(h)h'$ (because for a small change in height $dh$ the change in volume would be $dV = A(h)dh$, then divide by $dt$). The differential equation for $h$ becomes

$$A(h)h' = r_{in} - k\sqrt{h}$$  \hspace{1cm} (2.9)$$

Using these models we can predict when a tank will empty (that is, when $h(t) = 0$) as illustrated in the next example.

**Example 2.3.9** Consider a cylindrical can of soda with a small hole punched in the bottom. The radius of the can is 1 in and initially the soda in the can is 8 in deep. The soda begin to drain and after 10 sec the soda is 5 in deep. When will the can be empty?

**Solution.** We use the model in equation (2.8). We have $R = 1$, and there is no fluid coming into the can so $r_{in} = 0$. Thus the model simplifies to

$$h' = -\frac{k}{\pi}\sqrt{h}$$

This DE is separable. Using the method of separation of variables or software, a general solution is

$$h = \left( -\frac{k}{2\pi} t + C \right)^2$$

To find the values of $C$ and $k$, we use the conditions $h(0) = 8$ and $h(10) = 5$. The first condition gives $8 = C^2$ so $C = \sqrt{8}$. The second condition gives

$$5 = \left( -\frac{k}{2\pi} (10) + \sqrt{8} \right)^2 \implies k = \frac{\pi}{5} \left( \sqrt{8} - \sqrt{5} \right) \approx 0.3722.$$  

This yields the specific solution

$$h = \left[ -\frac{1}{10} \left( \sqrt{8} - \sqrt{5} \right) t + \sqrt{8} \right]^2 \approx h = (2.8284 - 0.059236t)^2.$$  

Setting $h = 0$ to find when the can is empty, we get

$$t = \frac{2.8284}{0.059236} \approx 47.75 \text{ sec.}$$
Exercises

Determine whether or not each equation is separable.

2.3.1 \( x' + 2x = e^{-t} \)
2.3.2 \( x' + 2x = 1 \)
2.3.3 \( x' = \frac{x + 1}{t + 1} \)
2.3.4 \( x' = \frac{\sin t}{\cos x} \)

Solve each below by the method of separation of variables:

2.3.5 \( x' = \frac{x}{t} \)
2.3.6 \( x' = \frac{t}{x} \)
2.3.7 \( x' = x + 5 \)
2.3.8 \( x' = 3x - 2 \)
2.3.9 \( x' = x \cos(t) \)
2.3.10 \( x' = (1 + t)(2 + x) \)

Solve each of the following initial-value problems:

2.3.11 \( y' = y + 1, \ y(0) = 2 \)
2.3.12 \( y' = ty, \ y(0) = 3 \)
2.3.13 \( x' = x \cos(t), \ x(0) = 1 \)
2.3.14 \( x' = (1 + t)(2 + x), \ x(0) = -1 \)
2.3.15 \( P' = 2P(1 - P), \ P(0) = 1/2 \)

2.3.16 (Population growth) A population \( P \) is growing according to the growth law \( \frac{dP}{dt} = rP \). Time \( t \) is measured in years. If the population is initially 100, and after 1 year the population is 150, how many will there be after 2 years? Hint: solve the differential equation for \( P \) as a function of \( t \). Then use the two conditions \( P(0) = 100 \) and \( P(1) = 150 \) to determine \( r \) and the integration constant. What happens to the population in the long term?

2.3.17 (Population growth) Repeat the previous problem, but this time assume the growth law \( \frac{dP}{dt} = rP(1 - P/300) \). What happens to the population in the long term?
2.3.18 (Falling bodies) In Section 1.1 we derived a DE to model a falling piece of crumpled paper. This model stated in terms of the velocity \( v \) is

\[ mv' = -mg - cv \]

where \( g = 9.8 \text{m/sec}^2 \) is the acceleration due to gravity, \( m \) is the mass in kg, and \( c \) is the constant of proportionality used to describe the force due to air resistance.

a. Assuming that both \( m \) and \( c \) are positive, find the general solution to the differential equation \( mv' = -mg - cv \) for \( v \) as a function of \( t \). (Hint: first show that \( \frac{dv}{g + cv} = -dt \).)

b. Find \( \lim_{t \to \infty} v(t) \). This is called the *terminal velocity* of the falling body.

c. Suppose that a body with mass 1 kg has a terminal velocity of \(-20 \text{m/sec}\). Find the value of \( c \).

d. According to this model, does the body ever actually reach terminal velocity? Briefly explain why or why not.

e. How long does it take the body from part c. to get within 1% of terminal velocity?

2.3.19 Suppose water is draining out of a tank through a hole in the bottom. At time \( t = 1 \) sec, the height of the water in the tank was 0.4 m and the volume of water in the tank was 1.1 m\(^3\). At time \( t = 1.5 \) sec, the height was 0.38 m and the volume was 1 m\(^3\). Use this information to estimate the value of \( k \) in equation (2.6). (Hint: \( V' \approx \text{(change in V)}/(\text{change in t}) \). Use an approximation of \( h \) in the right-hand side.)

2.3.20 Use equation (2.9) to find the model for \( h \) for the following types of tanks.

a. A cubical tank with sides of length \( l \).

b. A conical tank of height \( b \) and radius \( R \).

c. A spherical tank of radius \( R \).

2.3.21 In Example 2.3.9 we solved the DE \( h' = -\frac{k}{\sqrt{h}} \) where \( k \approx 0.3722 \) and got the specific solution \( h = (2.8284 - 0.059236t)^2 \). Show that this solution is not valid for \( t > 47.75 \). (Hint: Both sides of the DE must have the same sign.)

2.4 Numerical methods for first-order equations

In Section 1.2 we saw Euler’s method, which is a numerical method for approximating a solution to an IVP without having to find an explicit solution. Euler’s method is relatively easy to perform, but we have seen that the method can yield rather large errors, especially if a small step size is used. Fortunately, better numerical methods exist. These methods build on the ideas of Euler’s method and result in more complex algorithms. The increase in complexity is more than made up for in by an increase in speed and accuracy. In this section we discuss and compare two such methods.
Order of convergence

One way of comparing different numerical methods is with order of convergence. The next example illustrates some ideas related to the order of convergence.

**Example 2.4.1** Consider the DE \( y' = y - x \) with the initial condition \( y(0) = 0.5 \). The exact solution to this IVP is \( y = -0.5e^x + x + 1 \) (the reader should verify this) which yields the value \( y(2) = -0.69453 \). Use Euler’s method with step sizes of \( \Delta x = 0.5, 0.1, 0.01 \), and \( 0.001 \) to approximate the value of \( y(2) \).

**Solution.** Partial tables of the calculations from Euler’s method are shown below.

<table>
<thead>
<tr>
<th>( \Delta x = 0.5 )</th>
<th>( \Delta x = 0.1 )</th>
<th>( \Delta x = 0.01 )</th>
<th>( \Delta x = 0.001 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_i )</td>
<td>( y_i )</td>
<td>( x_i )</td>
<td>( y_i )</td>
</tr>
<tr>
<td>0.0</td>
<td>0.5</td>
<td>0.0</td>
<td>0.5</td>
</tr>
<tr>
<td>0.5</td>
<td>0.75</td>
<td>0.1</td>
<td>0.55</td>
</tr>
<tr>
<td>1.0</td>
<td>0.875</td>
<td>0.2</td>
<td>0.595</td>
</tr>
<tr>
<td>1.5</td>
<td>0.9125</td>
<td>1.9</td>
<td>-0.15795</td>
</tr>
<tr>
<td>2.0</td>
<td>0.46875</td>
<td>2.0</td>
<td>-0.36375</td>
</tr>
</tbody>
</table>

For \( \Delta x = 0.1 \), for example, we get the approximation \( y(2) \approx -0.36375 \). The difference between the exact value and the approximate value,

\[
| -0.69453 - (-0.36375) | = 0.33078
\]

is called the global truncation error. The global truncation errors for \( \Delta x = 0.05 \), \( \Delta x = 0.01 \), and \( \Delta x = 0.001 \) are 1.16328, 0.03652, and 0.00369, respectively.

Notice that in this example, as the step size gets smaller, the global truncation error also gets smaller. We could also say that as the step size gets smaller, the approximate solution converges to the exact solution. The order of convergence helps quantify this idea.

**Definition 2.4.1** The order of convergence of a numerical method is said to be \( k \) if the global truncation error is divided by approximately \( M^k \) when the step size is divided by \( M \) (for a sufficiently small step size). We also say the method is \( k \)th order convergent.
In Example 2.4.1, note that when the step size went from $\Delta x = 0.1$ to $\Delta x = 0.01$, the step size was divided by 10. The global truncation error went from 0.33078 to 0.03652, showing that the error was divided by about $10^1$. When the step size went from $\Delta x = 0.01$ to $\Delta x = 0.001$, we got similar results. This illustrates that Euler’s method is 1st order convergent.

However, note that when the step size went from $\Delta x = 0.5$ to $\Delta x = 0.1$, the step size was divided by 5 but the error was divided by about 3.5 (decreasing from 1.16328 to 0.33078). This shows the necessity of the qualifier “for a sufficiently small step size” in the definition of order of convergence. How small the step size needs to be is not easy to determine.

This example illustrates that in practical terms, 1st order convergent means that when the step size is small enough, reducing it by dividing by 10 yields one more digit of accuracy. Second order convergent means that the same reduction in step size will divide the error by $10^2 = 100$, meaning two more digits of accuracy. In general, $k$th order convergent means $k$ more digits of accuracy.

Runge-Kutta methods

In the early 1900’s, the German mathematicians C. Runge and M.W. Kutta developed two numerical methods that are widely used to this day. The first of these methods is called the Runge-Kutta second order method, or RK2 method for short. It is also known as the modified Euler method. This is a second order convergent method that is based on similar logic as Euler’s method.

**RK2 for First-Order IVP’s**

Given a differential equation $y' = f(x, y)$ and an initial condition $y(x_0) = y_0$, calculate the points $(x_1, y_1), \ldots, (x_n, y_n)$ using

\[
\begin{align*}
  k_1 &= f(x_i, y_i) \Delta x \\
  k_2 &= f \left( x_i + \frac{1}{2} \Delta x, y_i + \frac{1}{2} k_1 \right) \Delta x \\
  x_{i+1} &= x_i + \Delta x \\
  y_{i+1} &= y_i + k_2
\end{align*}
\]
for \( i = 0, \ldots, n - 1 \) where \( \Delta x \) is a small positive number called the step size.

The first step in RK2 along with the first step in Euler’s method are illustrated in Figure 2.10. We see that both methods use a straight line to approximate the change in the unknown function \( y(x) \) over the interval \((x_0, x_1)\). Euler’s method uses the line tangent to the solution curve at the point \((x_0, y_0)\). RK2 uses a line that goes through the midpoint of the Euler line. The RK2 line has slope \( f \left( x_0 + \frac{1}{2} \Delta x, y_0 + \frac{1}{2} k_1 \right) \).

![Figure 2.10: Illustration of RK2 method](image)

The other Runge-Kutta method we present here is a fourth order method called \( RK4 \). Many variations of this method are widely used in software packages. RK4 is based on approximating the change of the unknown function \( y \) over an interval by taking a weighted average of the slopes of the tangent lines at four different points in the interval: the beginning, twice in the middle, and once at the end.

**RK4 for First-Order IVP’s**

Given a differential equation \( y' = f(x, y) \) and an initial condition \( y(x_0) = y_0 \),
calculate the points \((x_1, y_1), \ldots, (x_n, y_n)\) using

\[
k_1 = f(x_i, y_i) \Delta x
\]

\[
k_2 = f\left(x_i + \frac{1}{2} \Delta x, y_i + \frac{1}{2} k_1\right) \Delta x
\]

\[
k_3 = f\left(x_i + \frac{1}{2} \Delta x, y_i + \frac{1}{2} k_2\right) \Delta x
\]

\[
k_4 = f\left(x_i + \Delta x, y_i + k_3\right) \Delta x
\]

\[
x_{i+1} = x_i + \Delta x
\]

\[
y_{i+1} = y_i + \frac{1}{6} k_1 + \frac{1}{3} k_2 + \frac{1}{3} k_3 + \frac{1}{6} k_4
\]

for \(i = 0, \ldots, n - 1\) where \(\Delta x\) is a small positive number called the step size.

The derivation of the weights \(1/6\) and \(1/3\) is beyond the level of this text. Suffice it to say that they are chosen using a Taylor series in order to ensure that the method is fourth-order convergent.

**Example 2.4.2** For the IVP \(y' = y - x, y(0) = 0.5\), use the RK2 and RK4 methods to estimate \(y(2)\) using a step size of \(\Delta x = 0.5\). Compare these estimate to the Euler estimate of 0.4688 found in Example 2.4.1, as well as to the exact value of \(y(2) = -0.6945\).

**Solution.** With \(f(x, y) = y - x, x_0 = 0, \) and \(y_0 = 0.5\), the first step in the RK2 method is

\[
k_1 = f(0, 0.5) \Delta x = (0.5 - 0)(0.5) = 0.25
\]

\[
k_2 = f\left(0 + \frac{1}{2}(0.5), 0.5 + \frac{1}{2}(0.25)\right) \Delta x = (0.625 - 0.25)(0.5) = 0.1875
\]

\[
x_1 = x_0 + \Delta x = 0 + 0.5 = 0.5
\]

\[
y_1 = y_0 + k_2 = 0.5 + 0.1875 = 0.6875
\]

The rest of the calculations are summarized in the Table 2.1. From the table we see that RK2 gives the approximation \(y(2) \approx -0.4865\).
The first step in the RK4 method is

\[
k_1 = f(0, 0.5)\Delta x = (0.5 - 0)(0.5) = 0.25
\]

\[
k_2 = f \left( 0 + \frac{1}{2} (0.5), 0.5 + \frac{1}{2} (0.25) \right) \Delta x = (0.625 - 0.25)(0.5) = 0.1875
\]

\[
k_3 = f \left( 0 + \frac{1}{2} (0.5), 0.5 + \frac{1}{2} (0.1875) \right) \Delta x = (0.5938 - 0.25)(0.5) = 0.1719
\]

\[
k_4 = f(0 + 0.5, 0.5 + 0.1719)\Delta x = (0.67188 - 0.5)(0.5) = 0.08594
\]

\[
x_1 = x_0 + \Delta x = 0 + 0.5 = 0.5
\]

\[
y_1 = y_0 + \frac{1}{6} (0.25) + \frac{1}{3} (0.1875) + \frac{1}{3} (0.1719) + \frac{1}{6} (0.0859) = 0.6758
\]

The rest of the values are summarized in the Table 2.2. From the table we see that RK4 gives the approximation \(y(2) \approx -0.6920\).

<table>
<thead>
<tr>
<th>(x)</th>
<th>(y)</th>
<th>(k_1)</th>
<th>(k_2)</th>
<th>(k_3)</th>
<th>(k_4)</th>
<th>(y + \frac{1}{6} k_1 + \cdots + \frac{1}{6} k_4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.5</td>
<td>0.25</td>
<td>0.1875</td>
<td>0.1719</td>
<td>0.0859</td>
<td>0.6758</td>
</tr>
<tr>
<td>0.5</td>
<td>0.6758</td>
<td>0.0879</td>
<td>-0.0151</td>
<td>-0.0409</td>
<td>-0.1826</td>
<td>0.6413</td>
</tr>
<tr>
<td>1.0</td>
<td>0.6413</td>
<td>-0.1793</td>
<td>-0.3492</td>
<td>-0.3916</td>
<td>-0.6252</td>
<td>0.2603</td>
</tr>
<tr>
<td>1.5</td>
<td>0.2603</td>
<td>-0.6198</td>
<td>-0.8998</td>
<td>-0.9698</td>
<td>-1.3547</td>
<td>-0.6920</td>
</tr>
<tr>
<td>2.0</td>
<td>-0.6920</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Table 2.2:** RK4 method for \(y' = y - x\)
CHAPTER 2  First-order Differential Equations

<table>
<thead>
<tr>
<th>Exact value</th>
<th>Euler estimate</th>
<th>RK2 estimate</th>
<th>RK4 estimate</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.6945</td>
<td>0.4688</td>
<td>-0.4865</td>
<td>-0.6920</td>
</tr>
</tbody>
</table>

Clearly the estimates are better with higher order methods.

Obviously RK2 and RK4 are very tedious to perform by hand. Fortunately these algorithms are built into most DE solving software, including the applets for this book. These software usually allow the user to choose the algorithm being used as well as the step size.

**Digits of accuracy**

Whenever we obtain a decimal approximation of a quantity, one question we should ask is, “How many digits are accurate?” Here is a suggested set of steps for answering this question and obtaining an estimate to an IVP accurate to a desired number of digits:

1. Choose an approximation method and a small step size. Use this to find an estimate.

2. Decrease the step size enough so that theoretically the number of digits of accuracy will increase by at least one. Use this step size to find a second estimate. Any digits that do not change from the first estimate to the second are assumed to be accurate.

3. If this number of digits of accuracy is not sufficient, repeat step 2.

In step 2, the necessary decrease in the step size depends on the order of convergence of the approximation method. To increase the number of digits of accuracy by at least one, we want to divide the error by at least 10. Euler’s method is first order, so dividing the step size by 10 should divide the error by 10. RK2 is second order, so dividing the step size by 4 should divide the error by \(4^2 = 16\). RK4 is fourth order, so dividing the step size by 2 should divide the error by \(2^4 = 16\). The next example illustrates this idea.

**Example** 2.4.3 For the initial value problem \(y' = y - x, \ y(0) = 0.5\), use the steps above to find a numerical estimate of \(y(2)\) which is accurate to four decimal places using Euler’s method, RK2, and RK4. Compare each estimate to the exact value \(y(2) = -0.6945\). For each method, start with step size 0.1. Use available software or the applet at [uhaweb.hartford.edu/rdecker/DeckerDEbook/DeckerDEbook.html](http://uhaweb.hartford.edu/rdecker/DeckerDEbook/DeckerDEbook.html) corresponding to this example to perform the calculations.

**Solution.** The results are summarized in the table below.
2.4 Numerical methods for first-order equations

<table>
<thead>
<tr>
<th></th>
<th>Euler</th>
<th></th>
<th>RK2</th>
<th></th>
<th>RK4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Δx</td>
<td>∆x</td>
<td>Δx</td>
<td>∆x</td>
<td>Δx</td>
<td></td>
</tr>
<tr>
<td>0.1</td>
<td>−0.36375</td>
<td>0.1</td>
<td>−0.68312</td>
<td>0.1</td>
<td>−0.69452</td>
</tr>
<tr>
<td>0.01</td>
<td>−0.65801</td>
<td>0.02</td>
<td>−0.69404</td>
<td>0.05</td>
<td>−0.69453</td>
</tr>
<tr>
<td>0.001</td>
<td>−0.69084</td>
<td>0.005</td>
<td>−0.69450</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.0001</td>
<td>−0.69416</td>
<td>0.0001</td>
<td>−0.69453</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.00001</td>
<td>−0.69449</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.00001</td>
<td>−0.69452</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

All three methods yield −0.6945 as the four decimal place approximation of \( y(2) \). Comparing this approximation to the exact value of −0.69453, we see that this approximation is indeed accurate to four digits. In fact, the RK2 and RK4 methods yield five decimal places of accuracy.

**Other numerical methods**

We have seen first, second, and fourth order convergent methods. The examples have demonstrated that as the order of convergence increases, the accuracy of the approximations also increases, but so does the complexity of the algorithm. Higher order convergent methods do exist, but at some point the increase in complexity is not compensated enough by the increase in accuracy to make the higher order method more efficient. Experience among experts in numerical analysis has shown that methods of order 4 or 5 tend to give the best results.

Some advanced numerical methods are modifications of RK4. One group of such methods is referred to as *adaptive step size methods*. These methods change the step size automatically in order to keep the error below a level determined by the user. Details of these methods are beyond the scope of this text. In software that use these methods, the user can adjust how the step size is changed with a setting called the *error tolerance*. The user should consult the software documentation for information on how the error tolerance works.

**Numerical Methods and DE Portraits**

In Section 2.2 we defined a portrait of a DE as a slope field together with a number of solution curves corresponding to different initial conditions. Software that draw solution curves do so by approximating points on the exact solution curves using numerical methods, and then connecting the dots. Therefore we need to be careful about step size (or error
tolerance) so that we get an accurate portrait. We can get idea of how accurate the portrait is by trying a smaller step size. If the portrait does not change much, then we assume the portrait is accurate.

**Example 2.4.4** Draw a portrait of the DE $x' = x + t$ over the region $-2 \leq t \leq 2, -2 \leq x \leq 2$ using available software. Use different step sizes to verify the phase portrait is accurate.

**Solution.** Figure 2.4.4 shows portraits generated with Euler’s method with step sizes of 0.5, 0.05, and 0.005.

Notice that the portrait with step size 0.5 is quite a bit different than then other portraits. The solution curves in this portrait touch, which violates Theorem 2.3. Thus we conclude that this portrait is very inaccurate. The portraits with step sizes 0.05 and 0.005 are very similar, so we conclude that they are sufficiently accurate.

**Exercises**

2.4.1 For the IVP $y' = -2y, \ y(0) = 1$, estimate the value of $y(2)$ by making a table of values using RK2 with 4 steps and step size $\Delta t = 0.5$ (do the work without using built-in computer or calculator methods, as in example 2.4.2). Repeat with RK4. Compare to the exact solution (which you can
calculate using separation of variables, or computer algebra) and give the amount of error for each method.

2.4.2 For the IVP \( x' = -x + e^t, \ x(0) = 2 \), estimate the value of \( x(1) \) by making a table of values using RK2 with 4 steps and step size \( \Delta t = 0.25 \) (do the work without using built-in computer or calculator methods, as in example 2.4.2). Repeat with RK4. Compare to the exact solution (which you can calculate using the integrating factor method, or computer algebra) and give the amount of error for each method.

For each I.V.P. 3 - 6, estimate the value of the dependent variable accurate to three significant digits at the point where the independent variable is equal to 5. First use Euler’s method, then repeat using either fourth-order Runge-Kutta or an adaptive step size RK method (as on the TI-89 calculator). Use a computer or calculator built-in method, or alter the first-order applet at .

2.4.3 \( y' = -y + \sin(t), \ y(0) = 1. \)

2.4.4 \( x' = x^2 - t, \ x(0) = 0. \)

2.4.5 \( y' = -0.1xy, \ y(0) = 4. \)

2.4.6 \( p' = p(1 - p) + 0.5 \cos(t), \ p(0) = 0. \)

Create accurate phase portraits of each differential equation 7 - 10, and include in the phase portrait the solution curve corresponding to the initial condition given. They are the same as the previous four problems, and you can use the same technology you used there.

2.4.7 \( y' = -y + \sin(t), \ y(0) = 1. \)

2.4.8 \( x' = x^2 - t, \ x(0) = 0. \)

2.4.9 \( y' = -0.1xy, \ y(0) = 4. \)

2.4.10 \( p' = p(1 - p) + 0.5 \cos(t), \ p(0) = 0. \)

2.5 Autonomous first-order equations and bifurcations

Differential equations always involve a dependent and at least one independent variable. In the DE

\[ y' = x + y, \]
for instance, the dependent variable is \( y \) and the independent variable is \( x \). Note that both of these variables appear explicitly in the DE. However, in the DE
\[
y' = 3y + 2,
\]
the dependent variable is \( y \), but the independent variable does not explicitly appear. In such cases, we typically choose either \( x \) or \( t \) as the name of the independent variable. The difference between these two DE’s leads to the following definition.

**Definition 2.5.1** A DE that does not dependent explicitly on the independent variable is called *autonomous*. An equation that is not autonomous is called *nonautonomous*.

A first-order autonomous DE has the general form
\[
y' = f(y),
\]
where \( f(y) \) is a function whose only variable is \( y \). Some DE’s may involve a parameter (a letter representing a constant quantity), but the presence of a parameter does not change whether the DE is autonomous or not. Parameters are typically denoted by letters other than \( x, y \), and \( t \).

**Example 2.5.1** In each DE below we identify the dependent variable, the independent variable, any parameters, and state whether or not the DE is autonomous. Note that in the second and fourth equations, no independent variable is explicitly given. In these cases we follow the convention that if the dependent variable is \( y \), we choose \( x \) as the independent variable; and if the dependent variable is \( x \), we choose \( t \) as the independent variable.

<table>
<thead>
<tr>
<th>Differential Equation</th>
<th>Dependent Variable</th>
<th>Independent Variable</th>
<th>Parameter(s)</th>
<th>Autonomous?</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y' = y + x )</td>
<td>( y )</td>
<td>( x )</td>
<td>none</td>
<td>no</td>
</tr>
<tr>
<td>( y' = 3 \sin(y) + y )</td>
<td>( y )</td>
<td>( x )</td>
<td>none</td>
<td>yes</td>
</tr>
<tr>
<td>( x' = at + x )</td>
<td>( x )</td>
<td>( t )</td>
<td>( a )</td>
<td>no</td>
</tr>
<tr>
<td>( x' = kx + b )</td>
<td>( x )</td>
<td>( t )</td>
<td>( k, b )</td>
<td>yes</td>
</tr>
</tbody>
</table>
Properties of autonomous DE’s

The fact that the right-hand side of the first-order autonomous DE \( y' = f(y) \) depends only on \( y \) and not the independent variable (which we will assume to be \( x \)) results in some special properties. Below we describe three such properties.

1. **Separability** A first-order autonomous DE is always separable because

\[
y' = f(y) \Rightarrow \frac{dy}{dx} = f(y) \Rightarrow \int \frac{1}{f(y)} \, dy = \int \, dx.
\]

This, however, does not mean that an explicit solution in terms of elementary functions is always obtainable.

2. **Slope Field** Along a horizontal line \( y = b \), where \( b \) is a constant, the slope marks in a slope field are identical. For example, consider the autonomous DE \( y' = y(1 - y) \). Along the line \( y = 2 \), for instance, all the slope marks have slope \( 2(1 - 2) = -1 \). A slope field of this DE is shown in Figure 2.12. Notice how the slope marks are identical along each “row” of marks.

![Slope field for \( y' = y(1 - y) \)](image)

**Figure 2.12:** Slope field for \( y' = y(1 - y) \)

3. **Constant Solutions** The constant function \( y(x) = a \) is a solution to the DE \( y' = f(y) \) if \( f(a) = 0 \). This is easily verified since the derivative of a constant function is 0. Constant solutions are also called *equilibrium solutions* and *fixed points*. Constant solutions do not always appear when a general solution is found using techniques such as separation of variables, so constant solutions must be found separately. Luckily this is relatively easy. All we need to do is set \( f(y) = 0 \) and solve for \( y \).
Example 2.5.2 Use separation of variables to find a general solution to each DE below. Also find any constant solution. For each constant solution, state whether or not it is part of the general solution that comes from separation of variables.

1. \( y' = ky \)
2. \( y' = ky(1 - y) \)
3. \( y' = e^{-y}(1 - y) \)

Solution. We omit most of the details of separation of variables. The reader may refer to Section 2.3 for a refresher on this technique.

1. Separation of variables yields the general solution \( y = Ce^{kx} \). To find constant solutions, we set \( ky = 0 \). This results in the constant solution \( y = 0 \). The constant solution corresponds to \( C = 0 \) in the general solution.

2. Separation of variables gives

\[
\int \frac{1}{y(1-y)} \, dy = \int k \, dx \quad \Rightarrow \quad \ln(y) - \ln(1-y) = kx + c \quad \Rightarrow \quad y = \frac{Ce^{kx}}{1 + Ce^{kx}}
\]

where \( C = e^c \). For constant solutions we set \( ky(1 - y) = 0 \), resulting in \( y = 0 \) and \( y = 1 \). The solution \( y = 0 \) corresponds to \( C = 0 \) in the general solution, but the \( y = 1 \) constant solution does not correspond to any finite value of \( C \).

3. Separation of variables gives

\[
\int \frac{e^y}{1-y} \, dy = \int dx.
\]

The integral on the left cannot be evaluated in terms of elementary functions, so no explicit general solution is possible. Setting \( e^{-y}(1 - y) = 0 \), we get the constant solution \( y = 1 \). \( \square \)

Graphical behavior of a solution

The constant solution(s) to an autonomous DE can tell us a great deal about the graphical behavior of a solution to the DE. To obtain this information, we use the idea of a zero of a function. A zero of a function \( f(x) \) is a number \( a \) such that \( f(a) = 0 \). We also use the following theorem, which is a result of the Intermediate Value Theorem of Calculus.
Theorem 2.4 If a function $f(x)$ is continuous, and if $f(x)$ has zeros at points $x = a$ and $x = b$ and no other zeros between $a$ and $b$, then $f(x)$ must be purely positive ($f(x) > 0$) or purely negative ($f(x) < 0$) for $a < x < b$. Similarly, if $f(x)$ has a zero $x = a$ but no other zero for $x > a$, then $f(x)$ must be purely positive or purely negative for $x > a$. A similar result holds if $f(x)$ has no other zero for $x < a$.

In the autonomous DE $y' = f(y)$, the zeros of $f(y)$ are the constant solutions. According to Theorem 2.4, for $y$ between two constant solutions, $f(y)$ must be purely positive or purely negative. The same is true for $y$ above the largest, or below the smallest constant solution. This is important for two reasons:

1. The sign of $f(y)$ is the same as the sign of $y'$.
2. If $y'$ is positive, then the graph of $y(x)$ is increasing. Likewise, if $y'$ is negative, then the graph of $y(x)$ is decreasing.

This result can be used to determine intervals over which the solution curves are increasing, and intervals on which the curves are decreasing. The next example illustrates this.

Example 2.5.3 For the differential equation $y' = y(y-2)^2(y-4)$, find all constant solutions. Then determine on which intervals of $y$ the solution curves are increasing, and on which intervals the solution curves are decreasing. From that information sketch a DE portrait.

Solution. Setting $f(y) = y(y-2)^2(y-4) = 0$ we get the constant solutions $y = 0$, $y = 2$, and $y = 4$. This divides the real number line into four intervals:

$$y < 0, \ 0 < y < 2, \ 2 < y < 4, \ \text{and} \ y > 4.$$ 

Since we know that the function $f(y)$ must be purely positive or purely negative on each interval, we can pick one $y$ value from each interval and substitute it into $f(y)$ to determine whether $f(y)$ is positive or negative on that interval.

For example, in the interval $y < 0$, we can choose $y = -1$ and evaluate

$$f(-1) = -1(-1 - 2)^2(-1 - 4) = 45,$$

which is positive. Thus we know that solution curves are increasing for $y < 0$. Results for the other intervals are shown in Table 2.3.

Figure 2.13 shows a DE portrait drawn with software. Notice that the behavior of solution curves agrees with the conclusions in Table 2.3 and that the horizontal curves correspond to constant solutions.
Analyzing the solution curves in Figure 2.13, we make the following observations:

1. Curves “near” the constant solution \( y = 0 \) approach this constant solution as \( x \) gets larger. In such a case we say that \( y = 0 \) attracts nearby solution curves.
2. Near the constant solution \( y = 4 \), solution curves move away as \( x \) gets larger. In this case we say that \( y = 4 \) repels nearby solution curves.
3. For the constant solution \( y = 2 \), solution curves just above \( y = 2 \) are attracted, and solution curves just below \( y = 2 \) are repelled.

These observations lead to the following definition.
**Definition 2.5.2** A constant solution that attracts nearby solution curves is called a *sink*. A constant solution that repels nearby solution curves is called a *source*. A constant solution that attracts solution curves on one side and repels them on the other side is called a *node*. We say that sinks are *stable*, sources are *unstable*, and nodes are *semi-stable*.

We can display information about an autonomous DE’s constant solutions, and the stability types of those solutions, using a graphical tool called a *phase line*. A phase line is a vertical number line with the constant solutions (or fixed points) marked on it. Between the fixed points, arrows are used to indicate which direction the solution curves move as $x$ increases (an up arrow for increasing and a down arrow for decreasing).

We can use the phase line to easily determine the stability of a constant solution.

- If the arrows on either side of a constant solution are pointing toward the constant solution, the solution is a sink.
- If the arrows are pointing away from the constant solution, the solution is a source.
- If one arrow points toward and one away from the constant solution, the solution is a node.

The next example illustrates these ideas.

**Example 2.5.4** Sketch a phase line for the DE $y' = y(y-2)^2(y-4)$ from Example 2.5.3, and use the phase line to classify the constant solutions as to their stability type (sink, source, or node). Also display a DE portrait next to the phase line. Finally, describe the long-term behaviors of the solutions corresponding to different values of the initial condition $y(0)$.

**Solution.** From Table 2.3, we see that the phase line should have the points $y = 0$, $y = 2$, and $y = 4$ marked as constant solutions. There should be up arrows for $y > 4$ and $y < 0$, and down arrows for $0 < y < 2$ and $2 < y < 4$. Figure 2.5.4 shows the phase line next to a DE portrait. From both the phase line and the portrait, we see that $y = 0$ is a sink, $y = 4$ is a source, and $y = 2$ is a node.

We can also see that if $y(0) > 4$ then $y \to \infty$ as $x$ increases, if $2 \leq y(0) < 4$ then $y \to 2$, and if $y(0) < 2$ then $y \to 0$. If $y(0)$ equals any of the constant solutions, then $y(t)$ does not change.
This example shows that a phase line and a portrait give similar information about a DE. They both show the constant solutions and they both show where solution curves are increasing and where they are decreasing. However, the phase line does not show how “quickly” each solution approaches its long-term behavior. For instance, in the portrait we see that for initial conditions greater than 4, $y(x)$ very quickly increases in value, as indicated by the nearly vertical slope lines. For initial conditions between 1 and 2, $y(x)$ approaches 0, but not very quickly, as indicated by the slope lines with a slightly negative slope. The phase line does not show this difference at all.

Bifurcations

A parameter is a constant within a DE. In many practical situations, we may not know the exact value of a parameter, or the parameter may change in value between one application of the DE and another. As we will see, the value of a parameter can affect the number of fixed points or their stability. Therefore, when analyzing a DE with a parameter, we need to analyze the fixed points for different values of the parameter. This leads to the following definition.

**Definition 2.5.3** In an autonomous DE involving a parameter $b$, we call $b^*$ a bifurcation point of the DE if the number of fixed points, or the stability type of any fixed point, changes when $b$ increases from $b < b^*$ to $b > b^*$. 
2.5 Autonomous first-order equations and bifurcations

Example 2.5.5 Consider the basic population model \( y' = ky \) where \( y \) denotes the size of a population at time \( t \). The parameter \( k \) describes how quickly the population grows. Suppose that for some animal species, \( k \) is proportional to the square of the population size so that \( k = ay^2 \) for some constant \( a \). This yields the model

\[
y' = ay^2.
\]

Now suppose that individuals emigrate (leave the population) at a constant rate \( b \geq 0 \). Using the rate of accumulation modeling principle, we end up with the model

\[
y' = ay^2 - b.
\]

Note that if we allow \( b \) to be negative, then the model would describe immigration (external addition of individuals).

For simplicity, assume that \( a = 1 \). Find all fixed points and create phase lines for the cases \( b > 0 \), \( b = 0 \), and \( b < 0 \) and label the stability type on each phase line. Determine any bifurcation points \( b^* \) and describe how the number of fixed points or their stability types change at \( b^* \). Finally, interpret these results in terms of the long-term behavior of the population.

Solution. The DE is \( y' = y^2 - b \), so to find fixed points we set \( y^2 - b = 0 \) resulting in \( y = \pm \sqrt{b} \). Thus for \( b > 0 \) there are two fixed points, for \( b = 0 \) there is one fixed point, and for \( b < 0 \) there is no fixed point. We analyze these three cases separately.

1. \( b < 0 \): The fixed points \( y = \pm \sqrt{b} \) are not real numbers, so there is no fixed point. Also \( y' = y^2 - b > 0 \) for all \( y \) so all arrows on the phase line point up.

2. \( b = 0 \): The DE is \( y' = y^2 \) and so the only fixed point is \( y = 0 \). Also, since \( y^2 \geq 0 \) for any \( y \), all arrows on the phase line point up. This makes \( y = 0 \) a node.

3. \( b > 0 \): Since there are two fixed points at \( y = \pm \sqrt{b} \) we need to determine the sign of \( y' \) on the intervals \( y > \sqrt{b} \), \(-\sqrt{b} < y < \sqrt{b} \), and \( y < -\sqrt{b} \). We can choose a value of \( y \) in each interval and calculate \( f(y) = y^2 - b \) at each chosen value as shown in Table 2.4. From the changes in the sign of \( f(y) \), we see that \( y = \sqrt{b} \) is a source and \( y = -\sqrt{b} \) is a sink.

Figure 2.15 shows phase lines for the three cases. Notice that as \( b \) increases, the number of fixed points changes from none for \( b < 0 \), to one for \( b = 0 \), to two for \( b > 0 \). Thus there is a bifurcation point at \( b^* = 0 \).

Next we analyze what these three phase lines tell us about the long-term behavior of the population relative to the initial population. Let \( y_0 \) denote the initial population.
Chapter 2  First-order Differential Equations

<table>
<thead>
<tr>
<th>Interval</th>
<th>y chosen</th>
<th>$f(y) = y^2 - b$</th>
<th>Sign of $f(y)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y &gt; \sqrt{b}$</td>
<td>$2\sqrt{b}$</td>
<td>$3b$</td>
<td>Pos</td>
</tr>
<tr>
<td>$-\sqrt{b} &lt; y &lt; \sqrt{b}$</td>
<td>$0$</td>
<td>$-b$</td>
<td>Neg</td>
</tr>
<tr>
<td>$y &lt; -\sqrt{b}$</td>
<td>$-2\sqrt{b}$</td>
<td>$b$</td>
<td>Pos</td>
</tr>
</tbody>
</table>

Table 2.4: Table of values for $y' = y^2 - b$ when $b > 0$

Figure 2.15: Phase lines for $y' = y^2 - b$ for $b < 0$, $b = 0$ and $b > 0$

1. $b < 0$: In this case individuals are immigrating. The upward pointing arrow indicates that the population will grow without bound ($y \to \infty$) regardless of the value of $y_0$.

2. $b = 0$: In this case no individual is immigrating or emmigrating. The upward pointing arrow for $y > 0$ indicates that for any $y_0 > 0$, the population will grow without bound. A negative value of $y$ does not make sense realistically, but mathematically the upward pointing arrow for $y < 0$ means that if $y_0 < 0$, then $y(x)$ will increase, approaching 0, as $x$ increases.

3. $b > 0$: In this case individuals are emmigrating. The upward pointing arrow for $y > \sqrt{b}$ indicates that for any $y_0 > \sqrt{b}$, the population will grow without bound. The fact that $\sqrt{b}$ is a fixed point means that if $y_0 = \sqrt{b}$, the population will stay at $\sqrt{b}$ forever. The downward pointing arrow between $\sqrt{b}$ and $-\sqrt{b}$ indicates that for any $y_0$ in this range, the population will approach $-\sqrt{b}$. In practical terms, this means that for any $y_0$ between 0 and $\sqrt{b}$, the population will go extinct.

In summary, these phase lines tell us that if there is immigration, the population will increase regardless of the initial population. If there is emigration, the population will increase if the initial population is large enough. If the initial population is small, then the population will die off.
Bifurcation diagrams

Figure 2.15 tells the story of how the fixed points of $y' = y^2 - b$ and their stability types depend on the parameter $b$. We can expand this figure into a type of graph called a bifurcation diagram, giving and even more detailed representation of a bifurcation. A bifurcation diagram is a graph of the fixed points of the DE vs the values of the parameter.

Bifurcation diagrams are hard to define in a few words. Instead of defining them, we give a bifurcation diagram for the DE $y' = y^2 - b$ in Figure 2.15, and then describe its important components. Diagrams for other DE’s contain similar components.

![Bifurcation Diagram](image)

**Figure 2.16**: Bifurcation diagram for $y' = y^2 - b$

The important components of this diagram include:

- The horizontal axis denotes values of the parameter $b$. The vertical axis denotes the value of the equilibrium solutions (or fixed points) corresponding to each value of $b$.
- The first quadrant contains a graph of $y = \sqrt{b}$. This is because for $b > 0$, one of the fixed points is $\sqrt{b}$. This curve is dashed to indicate that this fixed point is unstable.
- The fourth quadrant contains a graph of the other fixed point for $b > 0$, $y = -\sqrt{b}$. This curve is solid to indicate that this fixed point is stable.
- A typical phase line for $b > 0$ is shown on the right-half of the diagram.
- A phase line for $b = 0$ is shown on the vertical axis.
- The left-half of the diagram shows no graph of a fixed point because there is no fixed point for $b < 0$. Instead, the phase line for $b < 0$ is shown.
To interpret this diagram, we move along the horizontal axis starting on the left and make the following observations:

- For \( b < 0 \), there is no fixed point and the solution curves are always increasing.
- For \( b = 0 \), there is one fixed point at \( y = 0 \), and the solution curves are always increasing.
- For \( b > 0 \), there are two fixed points, one positive and one negative. The positive fixed point is unstable and the negative point is stable.
- Because the number of fixed points changes as \( b \) increases past 0, we conclude that there is a bifurcation at \( b^* = 0 \).

Exercises

In Exercises 1-10, draw a phase line for the autonomous differential equation and label each equilibrium point as a sink, source, or node.

2.5.1 \( x' = x(3 - x) \)
2.5.2 \( x' = x^4 - 1 \)
2.5.3 \( x' = x^2 \)
2.5.4 \( x' = (x^2 - 1)(x + 2)^2 \)
2.5.5 \( x' = \sin(x) \) (Hint: plot \( f(x) = \sin(x) \) to determine direction of the arrows.)
2.5.6 \( x' = x(a - x) \) for \( a > 0 \).
2.5.7 \( x' = x(a - x) \) for \( a < 0 \).
2.5.8 \( x' = x^2 - a \) for \( a > 0 \).
2.5.9 \( x' = x^2 - a \) for \( a < 0 \).
2.5.10 In Example 2.5.5, explain why if \( b = 0 \) and \( y_0 < 0 \), the \( y(x) \) cannot grow without bound even though the arrows on the phase line are all pointing upward. Specifically, explain why \( y(x) \) could never become positive in this case.
2.5.11 Sketch a bifurcation diagram for \( x' = x(a - x) \) using the values \( a = -2, -1, 0, 1, 2 \). What is the bifurcation point \( a^* \)? Briefly explain how the number and/or type of equilibrium point(s) change at the bifurcation point.
2.5.12 Sketch a bifurcation diagram for \( x' = x^2 - a \) using the values \( a = -2, -1, 0, 1, 2 \). What is the bifurcation point \( a^* \)? Briefly explain how the number and/or type of equilibrium point(s) change at the bifurcation point.
2.5.13 Sketch a phase line for \( x' = -\sqrt{|x(1-x)|} \). Now sketch a phase portrait by hand, based on the phase line. Then have a computer or calculator draw the solution curves in the phase portrait, using \( 0 \leq t \leq 5 \), \(-1 \leq x \leq 2 \). Is this different from what you expected? Does this contradict the uniqueness part of Theorem 2.2?
Lab 1: Population growth with harvesting

Introduction

A fish population is growing in a lake according to the model:

\[
\frac{dP}{dt} = aP \left(1 - \frac{P}{b}\right) - d
\]

where \( P \) denotes the population in thousands of fish after \( t \) years from some initial time. This is logistic growth with carrying capacity \( b \) and inherent growth rate \( a \). In addition there is fishing: \( d \) thousands of fish are being removed per year. For this lab, assume that \( a = 0.37 \) and \( b = 5.28 \).

Tools

You will need either computer software or calculator to do this lab. The applet for Lab 1 at \( \text{uhaweb.hartford.edu/rdecker/DeckerDEbook/DeckerDEbook.html} \) can be used for interactive exploration and to create the required graphs. This applet will not get exact/explicit solutions to differential equations, so if you want or require explicit solutions you will also need a computer algebra system, such as Maple, Mathematica, or the TI-89 calculator. Also, the Wolfram Alpha web site can solve differential equations; just type in the differential equation and it will try to solve it (get an explicit solution if possible) and supply some graphs.

Experiment

1. For \( d = 0 \) there is no fishing. Find the equilibrium values of the population for this case. Determine the type of stability at each equilibrium point. Produce a phase portrait (slope field plus numerous solution curves) for \(-1 \leq t \leq 10 \) and \(-2 \leq P \leq 8 \) and for values of \( P(0) \) ranging between \(-1 \) and \( 7 \) (include the equilibrium values). Choose your initial conditions carefully so as to get an accurate phase portrait. In your report, discuss the phase portrait, both in the context of the fish population and in purely mathematical terms. In your discussion, describe the long-term behavior of the population for various initial conditions.

2. Repeat part 1 for \( d = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7 \). Thus, for each value of \( d \) you need to find the equilibrium values, and create a phase portrait to include in your report. Be sure to include initial conditions corresponding to the equilibrium values.
for each case, so that the equilibrium solutions are plotted. In your report, group the $d$ values into groups that have similar behavior, and carefully describe what happens for various initial conditions within each group. Pay particular attention to the lower equilibrium value (when there is one), and explain what it represents in the context of fishing.

3. From part 2, you should see that there is a bifurcation point in terms of the parameter $d$ somewhere between $d = 0$ and $d = 0.7$. Based on your work in part 2, first estimate (based on the phase portraits and fixed points), and then find the exact value of the bifurcation point $\bar{d}$. In your report, describe how the number and/or type of equilibrium value(s) changes at $\bar{d}$. Sketch a bifurcation diagram to include in your report. What does this bifurcation point represent in terms of the amount of fishing and the fish population?

Hint on finding the exact bifurcation point: Solve for the equilibrium values as a function of $d$ (in other words, don’t substitute in a numerical value for $d$ when you set the right-hand side of the differential equation equal to zero). What would be the number of equilibrium values at the bifurcation point? Recall that quadratic equations have either two, one or no solutions depending on the quantity under the radical sign (called the discriminant) in the quadratic formula. Show all work in your report.

**Extensions**

4. Find an exact general solution for the differential equation using the numerical values for $d$ that you used in parts 1 and 2. You may want to use a computer algebra system for this. In your report, discuss how the form of the solution changes for different $d$ values (refer to the groups of $d$ values from part 2, and consider $d = 0$ to be a separate group). Use these formulas to determine the long-term behavior of the fish population for each $d$ value. To do this, eliminate terms that approach zero as $t \to \infty$. In particular, recall that $\lim_{t \to \infty} e^{kt} = 0$ for $k < 0$ and $\lim_{t \to \infty} e^{kt} = \infty$ for $k > 0$, $\lim_{t \to \frac{\pi}{2}} \tan(t) = \infty$, and $\tanh(t) = \frac{e^t - e^{-t}}{e^t + e^{-t}} = \frac{1-e^{-2t}}{1+e^{-2t}}$. You may have to do some algebra first to get the expression into a form that has terms that approach zero. In your report relate your results here to the results you got from the phase portraits parts 1) and 2). Does finite-time blow-up occur for any case? Could you see this from the phase portraits?

Note: Different computer algebra systems may give different forms for these general solutions. Identities that may help in showing the equivalence of the different forms are $\tan(ix) = i \tanh(x)$ and $\tanh(ix) = i \tan(x)$, where $i = \sqrt{-1}$ (the reader may
want to prove these). Also, complex values of the constant in the general solution may have to be considered to get all possible solutions.

5. Investigate the model

\[ \frac{dP}{dt} = aP \left(1 - \frac{P}{b}\right) - dP \]

For this model, the parameter \( d \) would represent the amount of fish removed each year per fish in the lake. This could be interpreted as proportional fishing, so that \( d = 0.5 \) would represent a fishing rate of 50% removed per year (fishing with large nets may result in a model such as this, as more fish would be removed from the same area when the population is larger). Use the approach outlined in parts 1-3 above. How do your results differ from the first model?

6. A model that includes the seasonal nature of fishing (more in the summer, less in the winter) is given by

\[ \frac{dP}{dt} = aP \left(1 - \frac{P}{b}\right) - d \left(1 - \cos(2\pi t)\right) \]

In this model there is no fishing at \( t = 0 \) or \( t = 1 \) (winter), and the greatest amount of fishing occurs at \( t = 0.5 \) (summer). Since this equation is nonautonomous, there are no equilibrium points. However, there may be periodic solutions, which we could also call equilibrium solutions. These are solution curves similar to the equilibrium solutions in parts 1-2, but they are cyclical (look like sin or cos curves) rather than constants. For each of the \( d \) values in part 2, plot the equilibrium solutions as best you can using numerical/graphical experimentation, and include a few other solution curves as well to form a phase portrait. There should be two equilibrium solutions for some \( d \) values, and none for others, just as in the autonomous case. One of the equilibrium solutions will behave like a sink, and the other like a source. The “source like” equilibrium solutions are hard to find when plotting forward in time, so try plotting backward in time.

At what time of year does the fish population peak for each equilibrium solution? Use \( d = 0.1 \). To do this, zoom in on each of the two equilibrium cycles. How is the time at which the population peaks related to the time at which the amount of fishing peaks?

Hard: Try to identify a bifurcation value \( \tilde{d} \) as in part 3). This would be the point where there is exactly one equilibrium solution. Since you can’t solve for equilibrium solutions as you do for equilibrium values, you can’t get an exact value for \( d \), so you
Chapter 3

Second-order Differential Equations

In this chapter we study second-order equations that can be put into the form

\[ y'' = f(x, y, y') \quad (3.1) \]

where \( f(x, y, y') \) denotes a function of the independent variable, the dependent variable, and the first derivative of the dependent variable. This function may include parameters, and the names of the independent and dependent variables may be different than \( x \) and \( y \), respectively.

One common application of second-order equations involves an object vibrating on a spring subject to damping as illustrated in Figure 3.1.

The object starts at rest position, the spring is compressed or stretched, the spring is released, and the object vibrates back and forth. The damper (also called a dashpot) resists the motion of the object. An external force may be applied to the object, further affecting the motion. This scenario describes, for example, a car hitting a bump and vibrating on its suspension. Let

\[ x(t) = \text{distance of the object from rest position at time } t \]

where \( x(t) > 0 \) indicates the object is to the right of the rest position and \( x(t) < 0 \) indicates the object is to the left of the rest position. Our goal is to describe this function \( x(t) \).

To model this situation, we use Newton’s second law, \( F = ma \) where \( F \) is the sum of all
the forces acting on the object, \( m \) is the mass of the object, and \( a \) is the acceleration of the object. There are three forces acting on the object.

1. **Force from the spring**: Hooke’s law for a spring states that the force of the spring is proportional to, and in the direction opposite of, the displacement from rest position (as long as the displacement is relatively small). In terms of \( x \), this law says

   \[ F_{\text{spring}} = -kx \]

   for some constant \( k > 0 \) called the *spring constant*. The spring constant measures the *stiffness* of the spring. The larger \( k \) is, the stiffer the spring.

2. **Force from the damper**: The damping force acts only when the mass is in motion and is assumed to be proportional to, and in the opposite direction of, the velocity. The velocity is described by \( x' \). This means that

   \[ F_{\text{damper}} = -cx' \]

   for some constant \( c > 0 \). The bigger \( c \) is, the stronger the damping force. Note that this model is not the only possible way to describe the damping force, but this model is the simplest. Also, it tends to be very accurate for slow moving objects.

3. **External force**: The external force may be constant, or may be changing over time (or may not exist at all). So we describe this force as a function of time:

   \[ F_{\text{external}} = f(t). \]
Now, let \( m \) denote the mass of the object. The acceleration of the object is described by \( x'' \). Newton’s second law yields:

\[
F_{\text{spring}} + F_{\text{damper}} + F_{\text{external}} = mx''.
\]

Substituting in the expressions for these forces yields:

\[
-kx - cx' + f(t) = mx''.
\]

Rearranging terms so the external force function is on the right yields the final model, called the mass-spring-damper equation:

\[
mx'' + cx' + kx = f(t).
\] (3.2)

If there is an external force (meaning \( f(t) \neq 0 \)), then we say the system is driven. If there is no external force, then \( f(t) = 0 \) and the model simplifies to

\[
mx'' + cx' + kx = 0.
\]

Some other examples of second-order equations that we will see throughout this chapter include:

1. **RLC circuit equation**
   \[
   LI'' + RI' + \frac{1}{C}I = V'(t)
   \]

2. **Duffing’s equation**
   \[
   x'' + \delta x' + \beta x + \alpha x^3 = \gamma \cos(\omega t)
   \]

3. **Van der Pol equation**
   \[
   x'' - \varepsilon(1-x^2)x' + x = 0
   \]

Note that all of these DE’s could be rewritten into the form of Equation (3.1). An example of a second order DE that cannot be put into the form of Equation (3.1) is

\[
y'' + y'' = y' + y + x.
\]

This is true because this equation cannot be solved explicitly for \( y'' \) in terms of \( y, y' \), and \( x \) (at least in terms of elementary functions).

Our approach to solving second-order equations will be very similar to solving first-order equations. We will classify the types of equations, and the type of the equation will dictate
the solution approach. Then we will find either an explicit, numerical, or graphical solution.

### 3.1 Linear Second-order Equations

In Chapter 2, we defined a linear first-order DE as an equation that can be written in the form $y' + g(x)y = h(x)$. We define a linear second-order equation in a similar way.

**Definition 3.1.1** A second-order DE is said to be linear if it can be written in the form

$$a(x)y'' + b(x)y' + c(x)y = f(x)$$

where $a(x)$, $b(x)$, $c(x)$, and $f(x)$ are functions of the independent variable. If $f(x) = 0$, the equation is said to be a homogeneous linear equation.

In the examples of second-order DE’s given in the introduction to this chapter, all but the last two equations (Duffing’s equation and the Van der Pol equation) are linear. We will often be able to find an explicit solution to a linear second-order equation. However, with nonlinear second-order equations, we will typically have to resort to numerical and graphical methods.

**Initial value problems**

An IVP involving a first-order equation consists of a first-order DE along with initial condition of the form $y(x_0) = y_0$. In Section 2.2 we proved a theorem about the interval on which a solution exists and it unique for linear DE’s. Second-order IVP’s satisfy a similar theorem, except that we need two initial conditions.

**Theorem 3.1** (Existence/Uniqueness for Linear Second-Order IVP’s) Consider an IVP of the form

$$a(x)y'' + b(x)y' + c(x)y = f(x), \quad y(x_0) = y_0, \quad y'(x_0) = v_0.$$ 

Assume that $b(x)/a(x)$, $c(x)/a(x)$, and $f(x)/a(x)$ are continuous on some interval $I$, and $x_0$ is in $I$. Then there exists a unique solution to the IVP that is defined on the interval $I$. 

□
A special type of linear second-order equation is one where the functions $a(x)$, $b(x)$, and $c(x)$ are all constant functions. Such an equation is of the form

$$ay'' + by' + cy = f(x)$$

where $a \neq 0$, $b$, and $c$ are constants and $f(x)$ is a function of the independent variable. This type of equation is called a *linear second-order DE with constant coefficients*. If $f(x)$ is continuous for all $x$, then Theorem 3.1 guarantees that a unique solution to an IVP involving this type of DE exists, and is defined for all $x$. Much of this section involves solving such an IVP.

**Linear Independence**

The theory behind solutions of linear second-order DE’s is based in part on the concept of *linearly independent functions*.

**Definition 3.1.2** Two nonzero functions $y_1(x)$ and $y_2(x)$ are said to be *linearly dependent* if there exists a nonzero constant $C$ such that

$$y_1(x) = Cy_2(x)$$

for all $x$. If the two functions are not linearly dependent, we say that they are *linearly independent*.

In simpler terms, two functions are linearly dependent if one is a constant multiple of the other. They are linearly independent if one is not a constant multiple of the other.

One important property of functions that are linearly *dependent* is that they have the same zeros. That is, if $a$ is a number such that $y_2(a) = 0$, then $y_1(a)$ also equals 0 since $y_1(a) = Cy_2(a) = C \cdot 0 = 0$. Graphically this means that both functions will have the same $x$-intercepts. If two functions have different zeros, then they must be linearly independent. The next example illustrates this idea.

**Example 3.1.1** Show that each of the following pairs of functions is linearly independent.

1. $y_1 = \sin(x)$ and $y_2 = \cos(x)$
2. \( y_1 = e^{2x} \) and \( y_2 = e^x \)

3. \( y_1 = x \) and \( y_1 = x^3 \)

**Solution.**

1. Note that \( \sin(0) = 0 \) and \( \cos(0) \neq 0 \). Thus these functions do not have the same zeros and are thus linearly independent.

2. If these functions were linearly dependent, then there would exist a constant \( C \neq 0 \) such that \( e^{2x} = Ce^x \) for all \( x \). Then at \( x = 0 \), we would have \( 1 = C(1) \) and so \( C = 1 \). At \( x = 1 \), we would have

\[
e^2 = Ce^1 \implies C = e^1 \approx 2.71.
\]

But \( C \) cannot equal 1 and 2.71 at the same time. So we have a contradiction. Hence the functions cannot be dependent, so they must be independent.

3. We could use a similar argument as in part 2 to prove independence. Another approach is to assume dependence, meaning assume there is a constant \( C \neq 0 \) such that \( x = Cx^3 \) for all \( x \). Taking the derivative of both sides with respect to \( x \) yields \( 1 = 3Cx^2 \). Taking the derivative again yields \( 0 = 6Cx \). Taking the derivative a third time yields \( 0 = 6C \), which means that \( C = 0 \). This is a contradiction, so we conclude that \( x \) and \( x^3 \) are linearly independent.

Similar arguments to those in Example 3.1.1 can be used to show that if \( m \neq n \) are nonzero constants, then the following pairs of functions are linearly independent:

1. \( \sin(mx) \) and \( \sin(nx) \)

2. \( \cos(mx) \) and \( \cos(nx) \)

3. \( e^{mx} \) and \( e^{nx} \)

4. \( x^m \) and \( x^n \)

The importance of linear independence will be seen in Theorem 3.3.
3.1 Linear Second-order Equations

Homogeneous linear equations with constant coefficients

We now turn our attention to the theory of solutions of second-order homogeneous linear equations with constant coefficients which have the general form.

\[ ay'' + by' + cy = 0 \]  \hspace{1cm} (3.3)

where \( a \neq 0, b, \) and \( c \) are constants. The next two theorems form the basis for this theory. They are stated in terms of second-order equations with constant coefficients, but they can be generalized to higher-order equations as well. They can also be generalized to equations with non-constant coefficients, under appropriate conditions.

**Theorem 3.2** If \( y_1(x) \) and \( y_2(x) \) each solve Equation (3.3), then

\[ y(x) = C_1 y_1(x) + C_2 y_2(x) \]

where \( C_1 \) and \( C_2 \) are constants also solves the equation.

**Proof.** The first and second derivatives of \( y(x) \) are

\[ y' = C_1 y_1' + C_2 y_2' \quad \text{and} \quad y'' = C_1 y_1'' + C_2 y_2'' \]

(we dropped the \( (x) \) for notational simplicity). Plugging \( y \) and its derivatives into Equation (3.3), and after some algebraic manipulation, we obtain

\[
ay'' + by' + cy = a \left( C_1 y_1'' + C_2 y_2'' \right) + b \left( C_1 y_1' + C_2 y_2' \right) + c \left( C_1 y_1 + C_2 y_2 \right)
= C_1 \left( ay_1'' + by_1' + cy_1 \right) + C_2 \left( ay_2'' + by_2' + cy_2 \right).
\]

But since \( y_1 \) and \( y_2 \) both satisfy Equation (3.3), the quantities in these last two sets of parentheses are both equal to 0. Thus we have

\[ ay'' + by' + cy = C_1(0) + C_2(0) = 0, \]

showing that \( y(x) \) also satisfies Equation (3.3).

The next theorem, which we present without proof, generalizes the results of Theorem 3.2.

**Theorem 3.3** If \( y_1(x) \) and \( y_2(x) \) each solve equation (3.3) and if these functions are linearly independent, then

\[ y(x) = C_1 y_1(x) + C_2 y_2(x) \]
where $C_1$ and $C_2$ are arbitrary constants is the general solution to equation (3.3). In this case, all possible solutions to the DE can be expressed in this form.

The expression $C_1 y_1(x) + C_2 y_2(x)$ is called a linear combination of the functions $y_1$ and $y_2$. Using this terminology, Theorems 3.2 and 3.3 can be summarized as saying:

1. Any linear combination of solutions to equation (3.3) is again a solution to the equation.
2. If we can find two linearly independent solutions to equation (3.3), then any other solution can be expressed as a linear combination of these solutions.

Since every solution to equation (3.3) can be expressed in the form $y(x) = C_1 y_1(x) + C_2 y_2(x)$, we will call this form “the” general solution to the equation (as opposed to “a” general solution). The next example illustrates these ideas for a rather simple DE.

**Example 3.1.2** Consider the DE $y'' = 0$.

a. Show that $y_1(x) = 2 + x$, $y_2(x) = 3$, and $y_3(x) = 4 + 5x$ are all solutions to the DE.

b. Show that $y_1$ and $y_2$ are linearly independent.

c. Find constants $C_1$ and $C_2$ such that $y_3(x) = C_1 y_1(x) + C_2 y_2(x)$.

**Solution.**

a. Simple calculus shows that $y_1'' = y_2'' = y_3'' = 0$. Thus these functions are all solutions to the DE.

b. Note that $y_1$ has a zero of $-2$ and $y_2$ has no zeros. Thus $y_1$ and $y_2$ are linearly independent.

c. We need to find $C_1$ and $C_2$ such that

$$4 + 5x = C_1(2 + x) + C_2(3) = (2C_1 + 3C_2) + C_1 x.$$ 

By equating corresponding coefficients from both sides of this equation, we get $C_1 = 5$ and $C_2 = -2$. 

\[\square\]
An Explicit Solution

Theorem 3.3 says that once we find a pair of linearly independent solutions $y_1$ and $y_2$, the general solution of equation (3.3) is completely determined. However, the theorem says nothing about how to find such functions. To find these functions, we begin with an educated guess. Very informally, equation (3.3) says that $y$ and constant multiples of its derivatives should all add up to 0. Thus we should expect $y$ and its derivatives to have the same form. An exponential function meets this requirement.

Thus we assume that a solution of (3.3) has the form $y = e^{rx}$ where $r$ is a constant. The first and second derivatives of $y$ are

$$y' = re^{rx} \quad \text{and} \quad y'' = r^2 e^{rx}.$$ 

Substituting these functions into DE (3.3) yields

$$ar^2 e^{rx} + bre^{rx} + ce^{rx} = 0.$$ 

Dividing by the non-zero function $e^{rx}$ yields the quadratic equation

$$ar^2 + br + c = 0. \tag{3.4}$$

This leads to the following definition.

**Definition** 3.1.3 For the DE $ay'' + by' + cy = 0$, the equation

$$ar^2 + br + c = 0$$

is called the *characteristic equation* and the polynomial $ar^2 + br + c$ is called the *characteristic polynomial*.

The solutions of the characteristic equation (or equivalently, the roots of the characteristic polynomial) are the values of the exponent needed for $e^{rx}$ to be a solution of equation (3.3). Therefore solving the differential equation (3.3) involves finding the roots of a quadratic polynomial. Using the quadratic formula, the roots of characteristic polynomial are

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$
The quantity \( d = b^2 - 4ac \) is called the discriminant. If \( d > 0 \), then there are two distinct real roots. If \( d = 0 \) then there is one repeated real root. If \( d < 0 \), then there are two complex roots. In each of these cases, we need to come up with two linearly independent solutions, then form a linear combination to get the general solution.

**Case 1, Real and distinct:** If the real numbers \( r_1 \) and \( r_2 \) are roots of the characteristic polynomial and \( r_1 \neq r_2 \), then both

\[
y_1(x) = e^{r_1x} \quad \text{and} \quad y_2(x) = e^{r_2x}
\]

are solutions to equation (3.3). By arguments similar to those in Example 3.1.1, these functions are linearly independent. Thus the general solution to equation (3.3) is

\[
y(x) = C_1 e^{r_1x} + C_2 e^{r_2x}.
\]

**Case 2, Real and repeated:** If \( r \) is the one real root of the characteristic polynomial, then \( y_1(x) = e^{rx} \) a solution to equation (3.3). It can also be shown that \( y_2(x) = xe^{rx} \) is also a solution (see Exercise 3.1.20). Since \( y_2 \) has a zero of 0 and \( y_1 \) has no zero, \( y_1 \) and \( y_2 \) are linearly independent. Thus the general solution to equation (3.3) is

\[
y(x) = C_1 e^{rx} + C_2 xe^{rx}.
\]

**Case 3, Complex:** If the roots of the characteristic polynomial are complex numbers, then we denote these roots \( r_1 = \alpha + \beta i \) and \( r_2 = \alpha - \beta i \) where \( \alpha \) and \( \beta \) are real numbers. To find solutions to equation (3.3), we use *Euler’s formula*

\[
e^{i\theta} = \cos(\theta) + i \sin(\theta)
\]

(see Exercise 3.1.22 for a derivation of this formula). The solutions to equation (3.3) are then

\[
e^{r_1x} = e^{(\alpha + \beta i)x} = e^{\alpha x} \cos(\beta x) + ie^{\alpha x} \sin(\beta x)
\]

and

\[
e^{r_2x} = e^{(\alpha - \beta i)x} = e^{\alpha x} \cos(\beta x) - ie^{\alpha x} \sin(\beta x).
\]

Both these solutions are complex-valued functions. To find real-valued solutions, we consider the following linear combinations of \( e^{r_1x} \) and \( e^{r_2x} \):

\[
y_1(x) = \frac{1}{2} e^{r_1x} + \frac{1}{2} e^{r_2x} = e^{\alpha x} \cos(\beta x)
\]
and

\[ y_2(x) = \frac{1}{2i} e^{r_1x} - \frac{1}{2i} e^{r_2x} = e^{\alpha x} \sin(\beta x). \]

By Theorem 3.2, each of these linear combinations is also a solution to equation (3.3). Next note that \[ y_1(0) = 1 \] and \[ y_2(0) = 0, \] showing that these functions have different zeros and are linearly independent. Thus the general solution to equation (3.3) in this case is

\[ y(x) = C_1 e^{\alpha x} \cos(\beta x) + C_2 e^{\alpha x} \sin(\beta x). \]

We summarize these three cases in Table 3.1.

<table>
<thead>
<tr>
<th>Roots</th>
<th>General Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>Two real roots ( r_1 \neq r_2 )</td>
<td>[ y(x) = C_1 e^{r_1x} + C_2 e^{r_2x} ]</td>
</tr>
<tr>
<td>One real root ( r )</td>
<td>[ y(x) = C_1 e^{rx} + C_2 x e^{rx} ]</td>
</tr>
<tr>
<td>Two complex roots ( \alpha \pm \beta i )</td>
<td>[ y(x) = C_1 e^{\alpha x} \cos(\beta x) + C_2 e^{\alpha x} \sin(\beta x) ]</td>
</tr>
</tbody>
</table>

**Table 3.1:** General Solutions to Linear Constant Coefficient Equations

**Example 3.1.3** Find a general solution to \[ y'' - 2y' - 3y = 0. \]

**Solution.** The characteristic polynomial is \( r^2 - 2r - 3 \), and the roots are

\[ r = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-3)}}{2(1)} = 3, -1. \]

This falls under the first case of two distinct real roots. Therefore, a general solution is

\[ y(x) = C_1 e^{3x} + C_2 e^{-x}. \]

**Initial value problems**

Theorem 3.1 guarantees that a solution to an IVP of the form

\[ ay'' + by' + cy = 0, \quad y(x_0) = y_0, \quad y'(x_0) = y_1 \]

exists and is unique on the interval \(-\infty < x < \infty\). To solve such an IVP, we first find the general solution using the results in Table 3.1, and then use the initial conditions to find the values of \( C_1 \) and \( C_2 \).
Example 3.1.4 Solve the IVP \( y'' - 4y' + 4y = 0, \ y(0) = 1, \ y'(0) = 3. \)

Solution. The characteristic polynomial is \( r^2 - 4r + 4 \) and the only root is \( r = 2 \). This falls under the second case of one real root. Therefore, a general solution is

\[
y(x) = C_1 e^{2x} + C_2 xe^{2x}.
\]

The initial condition \( y(0) = 1 \) gives

\[
1 = y(0) = C_1 e^{2(0)} + C_2(0)e^{2(0)} = C_1.
\]

Thus the solution has the form \( y(x) = e^{2x} + C_2 xe^{2x} \). To use the second initial condition, we find the derivative of this solution:

\[
y'(x) = 2e^{2x} + C_2 e^{2x} + 2C_2 xe^{2x}.
\]

The initial condition \( y'(0) = 3 \) then gives

\[
3 = y'(0) = 2 + C_2 + 0 \quad \Rightarrow \quad C_2 = 1.
\]

Thus the particular solution is

\[
y(x) = e^{2x} + xe^{2x}.
\]

Example 3.1.5 Solve the IVP \( y'' + 2y' + 26y = 0, \ y(0) = 1, \ y'(0) = -16. \)

Solution. The characteristic polynomial is \( r^2 + 2r + 26 \) and the roots are \( r = -1 \pm 5i \). This falls under the third case of complex roots. Therefore, a general solution is

\[
y(x) = C_1 e^{-x} \cos(5x) + C_2 e^{-x} \sin(5x).
\]

The initial condition \( y(0) = 1 \) yields

\[
1 = y(0) = C_1 e^{0} \cos(0) + C_2 e^{0} \sin(0) = C_1.
\]

The derivative of the solution is then

\[
y'(x) = -e^{-x} \cos(5x) - 5e^{-x} \sin(5x) + C_2(-e^{-x} \sin(5x) + 5e^{-x} \cos(5x)).
\]
The initial condition \( y'(0) = -16 \) yields
\[
-16 = y'(0) = -e^0 \cos(0) - 5e^0 \sin(0) + C_2(-e^0 \sin(0) + 5e^0 \cos(0)) = -1 + 5C_2
\]
\[
\Rightarrow \quad C_2 = -3.
\]
Thus the particular solution to the IVP is
\[
y(x) = e^{-x} (\cos(5x) - 3 \sin(5x)).
\]

\[\square\]

**Long-term behavior of solutions**

In many practical situations we are interested in the long-term behavior of a solution to a DE. That is, we want to know \( \lim_{x \to \infty} y(x) \). From Table 3.1, we see that if the DE is of the form \( ay'' + by' + cy = 0 \), then the general solution consists of two terms, each of which contains an exponential function of the form \( e^{rx} \) or \( xe^{rx} \). We can tell a lot about the long-term behavior of the solution using the following limits:

1. If \( r > 0 \), then \( \lim_{x \to \infty} e^{rx} = \infty \) and \( \lim_{x \to \infty} xe^{rx} = \infty \).
2. If \( r < 0 \), then \( \lim_{x \to \infty} e^{rx} = 0 \) and \( \lim_{x \to \infty} xe^{rx} = 0 \).

These limits mean that if the characteristic polynomial has a positive root (or complex roots with a positive real part), then the solution of the DE will grow without bound. If the roots are both negative (or complex roots with a negative real part), then the solution will approach 0. If the roots are complex, then the solution will oscillate with either increasing or decreasing amplitude. Figure 3.2 shows graphs of different possibilities.

To describe the long-term behavior of a solution, we really only need to know the roots of the characteristic polynomial. We do not necessarily need to know the values of the constants \( C_1 \) and \( C_2 \). To illustrate this idea, consider the scenario of an object vibrating on a spring as discussed in the introduction to this chapter. In this scenario, \( x(t) \) denotes the displacement of the object from rest position at time \( t \). If there is no external force, then \( x(t) \) is described by a DE of the form
\[
mx'' + kx + cx' = 0.
\]
Note that this is a second-order linear equation with constant coefficients. Initial conditions of the form \( x(t_0) = x_0 \), and \( x'(t_0) = x_1 \) describe the initial displacement and velocity of
CHAPTER 3  Second-order Differential Equations

\[ y = e^{rx}, \ r > 0 \]
\[ y = e^{rx}, \ r < 0 \]
\[ y = xe^{rx}, \ r > 0 \]
\[ y = xe^{rx}, \ r < 0 \]
\[ y = e^{x \cos \beta x}, \ \alpha > 0 \]
\[ y = e^{x \cos \beta x}, \ \alpha < 0 \]

Figure 3.2: Long-term behavior

the object, respectively. These initial conditions are necessary to find the values of the constants \( C_1 \) and \( C_2 \) in the general solution.

**Example 3.1.6** Suppose the motion of an object vibrating on a spring is described by the DE

\[ x'' + 2x' + 26x = 0. \]

Describe the long-term motion of the object.

**Solution.** Notice that this is the same DE as in Example 3.1.5 but with the independent and dependent variables of \( t \) and \( x \) rather than \( x \) and \( y \). In that example we showed that a general solution to the DE is

\[ x(t) = C_1 e^{-t} \cos(5t) + C_2 e^{-t} \sin(5t). \]

Notice that both the terms in this solution involve exponential functions with negative exponents. Thus \( \lim_{x \to \infty} x(t) = 0 \) regardless of the values of \( C_1 \) and \( C_2 \). In practical terms, this means that the object will eventually stop vibrating (at least asymptotically), and return to rest position, regardless of the initial displacement and initial velocity.

**Exercises**

**Directions:** Determine whether each of the following equations is linear or nonlinear. If it is linear state whether it is homogeneous or not.
3.1.1 \( y'' + x^2 y' + \sin(x) y = 0 \)
3.1.2 \( y'' + x^2 y' + x \sin(y) = 0 \)
3.1.3 \( x^2 y'' + xy' + y + x = 0 \)
3.1.4 \( t^2 x'' + tx' + 3x = 0 \)
3.1.5 \( x'' + x^2 = 0 \)
3.1.6 \( x'' + t^2 = 0 \)
3.1.7 In each part below, a DE and two functions \( y_1 \) and \( y_2 \) are given. Show that each function is a solution to the DE and that they are linearly independent. Then give the general solution to the DE.
   a. \( x^2 y'' - 2xy' + 2y = 0, x > 0, y_1 = x, y_2 = x^2 \)
   b. \( t^2 y'' + y = 0, t > 0, y_1 = \sqrt{t} \cos \left( \frac{1}{2} (\ln t) \sqrt{3} \right), y_2 = \sqrt{t} \sin \left( \frac{1}{2} (\ln t) \sqrt{3} \right) \)

Directions: Find a general solution to each of the following
3.1.8 \( x'' + 7x' + 10x = 0 \)
3.1.9 \( x'' - 3x' + x = 0 \)
3.1.10 \( x'' + 6x' + 9x = 0 \)
3.1.11 \( y'' - y' + \frac{1}{2} y = 0 \)
3.1.12 \( y'' + 2y' + 5y = 0 \)
3.1.13 \( x'' + x' + x = 0 \)

Directions: Find the solution to each of the following initial value problems, and describe the long-term behavior of each solution. Also, sketch a graph of each solution.
3.1.14 \( 2y'' + 3y' - 9y = 0, y(0) = 1, y'(0) = 0 \)
3.1.15 \( y'' + 0.3y' + 0.02y = 0, x(0) = 2, x'(0) = -5 \)
3.1.16 \( 4x'' - 4x' + x = 0, x(0) = 0, x'(0) = 1 \)
3.1.17 \( y'' - 2y' + 5y = 0, y(0) = 1, y'(0) = 1 \)
3.1.18 Use an approach similar to part 3 of Example 3.1.1 to show that if \( 0 = Ax^3 + Bx^2 + Cx + D \) for all \( x \), then \( A = B = C = D = 0 \). Then generalize this argument to show that if
\[
0 = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0
\]
for all \( x \) where \( a_n, \ldots, a_0 \) are constants, then \( a_n = \cdots = a_0 = 0 \).
3.1.19 For each of the three values \( c = 0.22, 0.23, \) and \( 0.24 \), solve the IVP

\[
x'' + cx' + 0.013225x = 0, \quad x(0) = 2, \quad x'(0) = 0.
\]

Explain what is happening as the value of \( c \) increases.

3.1.20 For the DE \( ay'' + by' + cy = 0 \), if \( b^2 - 4ac = 0 \), then \( y = x e^{rx} \) is another solution to the DE. (Hint: Substitute \( y = x e^{rx} \) into the DE and use the fact that \( r = -b/(2a) \) and that \( y = e^{rx} \) is a solution.)

3.1.21 In Example 3.1.5 we claimed that a general solution of \( y'' + 2y' + 26y = 0 \) is

\[
y(x) = C_1 e^{-x} \cos(5x) + C_2 e^{-x} \sin(5x).
\]

Verify that this is so by showing that \( y_1(x) = e^{-x} \cos(5x) \) and \( y_2(x) = e^{-x} \sin(5x) \) are solutions to the DE.

3.1.22 In this exercise we will derive Euler's formula, \( e^{i\theta} = \cos(\theta) + i \sin(\theta) \), using the Maclaurin series for \( e^x, \sin x, \) and \( \cos x \).

\[
e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots, \quad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots, \quad \text{and} \quad \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots,
\]

a. Evaluate \( e^{i\theta} \) by plugging \( i\theta \) in for \( x \) in the Maclaurin series for \( e^x \). Write out at least the first six terms of this series.

b. Simplify the terms in part a. using the fact that \( i^2 = -1 \).

c. From part b., collect the terms involving \( i \) and those that are real, and arrive at

\[
e^{i\theta} = \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \cdots\right) + i \left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots\right)
\]

d. What is the quantity in the first set of parentheses in terms of \( \sin \) or \( \cos \)? What about the second set? Arrive at the conclusion \( e^{i\theta} = \cos(\theta) + i \sin(\theta) \).

**Introduction to Exercises 3.1.23 and 3.1.24**

One way to determine independence of two functions \( y_1(x) \) and \( y_2(x) \) is with a function called the *Wronskian*, named after the Polish mathematician H. Wronski (1778-1863):

\[
W[y_1, y_2](x) = y_1(x)y_2'(x) - y_1'(x)y_2(x).
\]

The Wronskian can also be written as the determinate of a \( 2 \times 2 \) matrix:

\[
W[y_1, y_2](x) = \begin{vmatrix}
y_1(x) & y_2(x) 
y_1'(x) & y_2'(x)
\end{vmatrix}.
\]
3.1.23 Show that if \( y_1(x) \) and \( y_2(x) \) are linearly dependent, meaning that \( y_1(x) = cy_2(x) \) for some constant \( c \), then \( W[y_1, y_2](x) = 0 \) for all \( x \).

3.1.24 Suppose that \( y_1(x) \) and \( y_2(x) \) are both solutions to the DE \( ay'' + by' + cy = 0 \) where \( a, b, \) and \( c \) are constants and \( a \neq 0 \). Let \( x_0 \) be a number. Suppose the following three statements are true:

1. \( W[y_1, y_2](x_0) = y_1(x_0)y_2'(x_0) - y_1'(x_0)y_2(x_0) = 0 \)
2. \( y_1(x_0) \neq 0 \) and \( y_2(x_0) \neq 0 \)
3. \( y_2 \) satisfies the initial conditions \( y_2(x_0) = y_0 \) and \( y_2'(x_0) = v_0 \) where \( y_0 \) and \( v_0 \) are numbers

Solve the following problems related to this scenario.

a. Let \( k = \frac{y_2(x_0)}{y_1(x_0)} \) and define the function \( y_3(x) = ky_1(x) \). Show that \( y_3(x) \) is a solution to the DE \( ay'' + by' + cy = 0 \).

b. Show that \( y_3(x) \) satisfies the initial conditions \( y_3(x_0) = y_0 \) and \( y_3'(x_0) = v_0 \).

c. Use Theorem 3.1 to show that \( y_3(x) = y_2(x) \) for all \( x \).

d. Use part c. to show that \( y_1(x) \) and \( y_2(x) \) are linearly dependent.

3.2 Harmonic Oscillators

In this section we focus on analyzing the motion of an object vibrating on a spring as described in the introduction to this chapter. Such a system involves an object with mass \( m > 0 \), a spring with a spring constant \( k > 0 \), and a damper (or dashpot) with damping coefficient \( c \geq 0 \). If there is no damper, then \( c = 0 \) and the system is said to be undamped. The system of the object and the spring together is called a harmonic oscillator. In this section we do not consider any external forces. This system is illustrated in Figure 3.3.

If \( x(t) \) denotes the displacement of the object from rest position, then as shown in the introduction to this chapter, the DE describing \( x \) is

\[
mx'' + cx' + kx = 0.
\]

This is just a second order, linear, constant coefficient, homogeneous DE. The solution to this DE is called the equation of motion. As shown in Section 3.1, this solution depends on the roots of the characteristic polynomial \( mr^2 + cr + k \),

\[
r = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}.
\]
Undamped systems

If the system is undamped (meaning \( c = 0 \)), then the roots of the characteristic polynomial are

\[
    r = \pm \frac{\sqrt{-4mk}}{2m} = \pm \frac{\sqrt{4mk}}{2m} \ i = \pm \sqrt{\frac{k}{m}} \ i.
\]

Now let \( \beta = \sqrt{k/m} \). Then the general solution to the DE is

\[
    x(t) = C_1 \cos(\beta t) + C_2 \sin(\beta t).
\]

We can write this general solution in a slightly different way using the trigonometric identity

\[
    A \sin(\beta t - \phi) = -A \sin \phi \cos(\beta t) + A \cos \phi \sin(\beta t).
\]

If we choose the number \( \phi \) such that \( C_1 = -A \sin \phi \) and \( C_2 = A \cos \phi \), then \( \phi \) must also satisfy

\[
    \frac{C_1}{C_2} = \frac{-A \sin \phi}{A \cos \phi} = -\tan \phi
\]

and \( A \) must satisfy

\[
    C_1^2 + C_2^2 = (-A \sin \phi)^2 + (A \cos \phi)^2 = A^2 \left( \sin^2 \phi + \cos^2 \phi \right) = A^2.
\]

We summarize these arguments in the following theorem.

**Theorem 3.4** In an undamped system where the object has mass \( m \) and the spring has
stiffness $k$, the equation of motion has the form

$$x(t) = C_1 \cos(\beta t) + C_2 \sin(\beta t) = A \sin(\beta t - \phi)$$

where $\beta = \sqrt{k/m}$, $A = \sqrt{C_1^2 + C_2^2}$, and $\phi$ satisfies $\tan(\phi) = -C_1/C_2$ and $C_1 = -A \sin \phi$. The values of $C_1$ and $C_2$ are determined by the initial conditions.\footnote{We can calculate $\phi$ by $\phi = \arctan(-C_1/C_2)$ or $\phi = \arctan(-C_1/C_2) + \pi$, whichever satisfies $C_1 = -A \sin \phi$.}

Figure 3.4 shows a typical graph of the equation of motion $x(t) = A \sin(\beta t - \phi)$. Note that the object vibrates with constant oscillations. Also note that the object is at rest position ($x = 0$) at time $t = \phi/\beta$, moves to the right ($x > 0$), returns to the rest position, moves to the left ($x < 0$), and then returns to the rest position again. This is called a cycle. This leads to the following definitions.

**Definition 3.2.1** For an undamped system whose equation of motion is

$$x(t) = A \sin(\beta t - \phi),$$

the object completes one cycle when it begins at rest position, moves in one direction, returns to rest position, moves in the opposite direction, and then returns to rest position again.

- The period is $2\pi/\beta$ and describes the amount of time necessary to complete one cycle.
- The frequency is $1/\text{period} = \beta/(2\pi)$ and describes the number of cycles in one unit of time.
- The amplitude is $A$ and describes how “big” the oscillations are.
- The phase shift is $\phi/\beta$ and describes how far to the right (in units of time) the graph of the equation of motion is shifted compared to the graph of $\sin t$.

If time is measured in seconds (s), then the period has units of seconds and the frequency has units of hertz (cycles/s), abbreviated Hz.
Notice from Figure 3.4 that the period is also the amount of time between consecutive high points (or low points) on the graph of the equation of motion. Also, we could think of a cycle occurring from one high point (or low point) to the next.

**Units and initial conditions**

By the balance of units principle, we need each term in the DE $m x'' + c x' + k x = 0$ to have the same units. In this text we will use the convention of measuring distance in meters (m), time in seconds (s), and mass in kilograms (kg). A Newton (N) is an amount of force that will accelerate a mass of 1 kg at a rate of 1 m/s$^2$. Since force equals mass times acceleration, the term $m x''$ represents force and its units are $\text{kg} \cdot \text{m} / \text{s}^2 = \text{N}$.

For the other terms in the DE $m x'' + c x' + k x = 0$ to have the units of N, we need the constant $c$ to have the units

$$\frac{\text{N}}{\text{m/s}} = \frac{\text{N} \cdot \text{s}}{\text{m}} = \frac{\text{kg}}{\text{s}}$$

and the constant $k$ to have the units

$$\frac{\text{N}}{\text{m}} = \frac{\text{kg}}{\text{s}^2}.$$
3.2 Harmonic Oscillators

In physical terms, these initial conditions give the initial displacement and velocity, respectively, of the object. A positive value of $x(0) = x_0$ means the object starts to the right of its rest position. A positive value of $x'(0) = x_1$ means the object is initially moving to the right. Negative values mean the opposite, while values of 0 mean the object begins at rest position or is initially not moving, respectively. The initial conditions could also be given at a more generic time $t = t_0$ rather than $t = 0$.

**Example 3.2.1** An object with a mass of 3 kg attached to a spring with a stiffness of 48 N/m is initially displaced 0.5 m to the left of the rest position and given an initial velocity of 2 m/s to the right. If there is no damping, find the equation of motion, the amplitude, the period, the frequency, and the phase shift.

**Solution.** We are given that $m = 3$, $k = 48$, and $c = 0$, so that DE we want to solve is $3x'' + 48x = 0$. The characteristic polynomial is $3r^2 + 48$ and its roots are

$$r = -0 \pm \frac{\sqrt{0^2 - 4(3)(48)}}{2(3)} = \pm \frac{\sqrt{-576}}{6} = \pm 4i$$

The general solution to the DE is $x(t) = C_1 \cos(4t) + C_2 \sin(4t)$. To find $C_1$ and $C_2$, we use the given initial conditions $x(0) = -0.5$ and $x'(0) = 2$. The first initial condition gives

$$-0.5 = x(0) = C_1 \cos(0) + C_2 \sin(0) = C_1.$$  

The derivative of $x$ is then

$$x'(t) = 2 \sin(4t) + 4C_2 \cos(4t).$$

The condition $x'(0) = 2$ gives

$$2 = x'(0) = 2 \sin(0) + 4C_2 \cos(0) = 4C_2 \implies C_2 = 0.5.$$  

Thus the equation of motion is

$$x(t) = -0.5 \cos(4t) + 0.5 \sin(4t).$$

Next we use Theorem 3.4 to describe the equation of motion in the form $x(t) = A \sin(\beta t - \phi)$. We have

$$A = \sqrt{C_1^2 + C_2^2} = \sqrt{(-0.5)^2 + (0.5)^2} \approx 0.707.$$
To find $\phi$, we first try

$$\phi = \arctan \left( \frac{-C_1}{C_2} \right) = \arctan \left( \frac{-(-0.5)}{0.5} \right) = \arctan(1) = \frac{\pi}{4} \approx 0.785$$

and observe that

$$-A \sin \phi \approx -0.707 \sin(0.785) \approx -0.5.$$ 

Since this quantity equals $C_1$, we conclude that $\phi = 0.785$ is an appropriate value. Thus we can describe the equation of motion as

$$x(t) = 0.707 \sin(4t - 0.785).$$

Therefore, the amplitude is $A = 0.707$, the period is $2\pi/\beta \approx 1.571$, the frequency is $1/1.571 \approx 0.637$, and the phase shift is $\phi/\beta \approx 0.196$. A graph of this equation of motion is shown in Figure 3.5.

![Graph of $x(t) = 0.707 \sin(4t - 0.785)$](image)

**Figure 3.5:** Graph of $x(t) = 0.707 \sin(4t - 0.785)$

---

**Damped vibrations**

If the system is damped (meaning $c > 0$), then the behavior of the equation of motion is determined by the sign of the discriminate $c^2 - 4mk$. We consider three cases: positive, zero, and negative.

**Over-damped:** $c^2 - 4mk > 0$
In this case, the roots of the characteristic polynomial are

\[ r_1 = \frac{-c + \sqrt{c^2 - 4mk}}{2m} \quad \text{and} \quad r_2 = \frac{-c - \sqrt{c^2 - 4mk}}{2m}. \]

Since \( c^2 - 4mk > 0 \), both these roots are real. Furthermore, it can be shown that both of these roots are negative (see Exercise 3.2.11). The equation of motion has the form

\[ x(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}. \]

The fact that there is no sine or cosine term in the solution indicates that the object will not oscillate. Since both the terms in the equation are exponential with negative exponents, \( x(t) \) rather quickly approaches 0. This means that the object will asymptotically approach its rest position and come to a stop. We call the system over-damped.

**Critically-damped: \( c^2 - 4mk = 0 \)**

In this case, the root of the characteristic polynomial is

\[ r = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m} = \frac{-c}{2m}, \]

which is negative. The equation of motion has the form

\[ x(t) = C_1 e^{rt} + C_2 t e^{rt}. \]

Again, the fact that is no sin or cos term in the solution indicates that the object will not oscillate. The exponential terms with negative exponents mean that and \( x(t) \) approaches 0, as in the over-damped case. In this case we call the system critically-damped.

Notice that the equations of motion for over- and critically-damped systems are very similar. Their graphs also look very similar. Figure 3.6 shows two examples of what these graphs could look like. Either type of system could have either graph. Note that these graphs either cross the \( t \)-axis exactly once, or not at all. This is true in general for over- and critically-damped systems (see Exercise 3.2.13 and 3.2.14).

**Under-damped: \( c^2 - 4mk < 0 \)**

In this case, the roots of the characteristic polynomial can be written as

\[ r = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m} = \frac{-c}{2m} \pm \frac{\sqrt{4mk - c^2}}{2m} i. \]
Let

\[ \alpha = -\frac{c}{2m} \quad \text{and} \quad \beta = \frac{\sqrt{4mk - c^2}}{2m}, \]

Then we can write these roots as \( \alpha \pm \beta i \). Note that the real part, \( \alpha \), is negative. The

equation of motion has the form

\[ x(t) = e^{\alpha t} \left[ C_1 \cos(\beta t) + C_2 \sin(\beta t) \right] \]

Notice that the quantity in the square brackets is the same as the equation of motion for an undamped system. Thus by Theorem 3.4, this equation of motion can alternatively be written as

\[ x(t) = Ae^{\alpha t} \sin(\beta t - \phi) \]

where \( A \) and \( \phi \) satisfy the same conditions as in Theorem 3.4. However, \( \beta \) is calculated differently. This alternate form of the equation of motion shows two characteristics of the motion:

1. The object will oscillate to the left and right of the rest position due to the \( \sin \) term.

2. The oscillations will decrease in amplitude due to the coefficient of \( Ae^{\alpha t} \) which approaches 0 as \( t \) increases due to the negative exponent. This decreasing amplitude means that in the long-run, \( x(t) \) will approach 0 as in the over- and critically-damped cases.

In this case we call the system \textit{under-damped}. A cycle of an under-damped system can be defined in a way similar to that of an undamped system (see Definition 3.2.1) by using the...
zeros of the solution. The period and frequency can then be defined in the same way as for the undamped system (some texts may refer to the *quasi-period* in the case of under-damped motion since the motion is not, strictly speaking, periodic). A typical graph of the equation of motion of an under-damped system is shown in Figure 3.7.

![Under-Damped Motion](image)

**Figure 3.7:** Typical under-damped motion

**Example 3.2.2** A 1 kg object on a spring is displaced 0.1 m to the right of its rest position and given a push in the opposite direction resulting in an initial velocity of $-0.2$ m/s. The spring constant is 1 N/m and there is damping with a constant of 2 N·s/m. Find the equation of motion of the object, sketch a graph of the motion for $0 \leq t \leq 6$, and describe the long-term behavior.

**Solution.** We are given $m = 1$, $k = 1$, and $c = 2$, so that DE we want to solve is $x'' + 2x' + x = 0$. The characteristic polynomial is $r^2 + 2r + 1$ and its roots are

$$r = \frac{-2 \pm \sqrt{2^2 - 4(1)(1)}}{2(1)} = \frac{-2 \pm \sqrt{0}}{2} = -1.$$  

Since the discriminate is 0, the system is critically-damped. The general solution to the DE is $x(t) = C_1 e^{-t} + C_2 te^{-t}$. In terms of $x(t)$, the initial conditions are given as

$$x(0) = 0.1 \quad \text{and} \quad x'(0) = -0.2.$$  

The condition $x(0) = 0.1$ means that

$$0.1 = x(0) = C_1 e^0 + C_2(0) = C_1.$$
The derivative of the general solution is then

\[ x'(t) = -0.1e^{-t} + C_2e^{-t} - C_2 te^{-t} \]

The condition \( x'(0) = -0.2 \) means that

\[-0.2 = x'(0) = -0.1e^0 + C_2e^0 - C_2(0) = -0.1 + C_2 \quad \Rightarrow \quad C_2 = -0.1. \]

Thus the equation of motion is

\[ x(t) = 0.1e^{-t} - 0.1te^{-t}. \]

The exponential terms with negative exponents mean that \( x(t) \) asymptotically approaches zero as \( t \) gets large so that the mass returns to its rest position without oscillating. The graph of the solution is given in Figure 3.8. The figure shows that the object crosses the rest position at \( t = 1 \) s before asymptotically coming to rest.

![Graph of \( x(t) = 0.1e^{-t} - 0.1te^{-t} \)](image)

**Figure 3.8:** Graph of \( x(t) = 0.1e^{-t} - 0.1te^{-t} \)

Notice that in Example 3.2.2, the object crosses its rest position exactly once. It can be shown that in the critically-damped or over-damped cases there can be at most one point in time when the object crosses its rest position (see Exercises 3.2.14 and 3.2.13).

**Example 3.2.3** A 1 kg object on a spring is displaced 0.5 m to the right of its rest position and given a push to the right resulting in an initial velocity of 0.8 m/s. The spring constant is 6 N/m and there is damping with a constant of 4 N·s/m. Find the equation of motion of the object, sketch a graph of the motion, and find the period, frequency, and phase shift.
Solution. We are given \( m = 1, \ k = 6, \) and \( c = 4, \) so that DE we want to solve is \( x'' + 4x' + 6x = 0. \) The characteristic polynomial is \( r^2 + 4r + 6 \) and its roots are

\[
r = \frac{-4 \pm \sqrt{4^2 - 4(1)(6)}}{2(1)} = \frac{-4 \pm \sqrt{-8}}{2} = -2 \pm \sqrt{2}i.
\]

Since the discriminate is negative, the system is under-damped. The general solution to the DE is

\[
x(t) = e^{-2t} \left[ C_1 \cos(\sqrt{2}t) + C_2 \sin(\sqrt{2}t) \right].
\]

In terms of \( x(t) \), the initial conditions are given as

\[
x(0) = 0.5 \quad \text{and} \quad x'(0) = 0.8.
\]

Using these initial conditions to solve for \( C_1 \) and \( C_2 \) yield \( C_1 = 0.5 \) and \( C_2 = 1.8/\sqrt{2} \approx 1.273 \) (the reader should verify this). Therefore, the equation of motion is

\[
x(t) = e^{-2t} \left[ 0.5 \cos(\sqrt{2}t) + 1.273 \sin(\sqrt{2}t) \right].
\]

To find the period, frequency and phase shift, we first calculate

\[
A = \sqrt{C_1^2 + C_2^2} = \sqrt{(0.5)^2 + (1.273)^2} \approx 1.367.
\]

To find \( \phi \), we first try

\[
\phi = \arctan \left( \frac{-C_1}{C_2} \right) = \arctan \left( \frac{-0.5}{1.273} \right) \approx -0.374
\]

and verify that \( -A \sin \phi \approx -1.367 \sin(-0.374) \approx 0.5 = C_1 \). Thus we can describe the equation of motion as

\[
x(t) = 1.367e^{-2t} \sin(\sqrt{2}t + 0.374).
\]

The period is \( 2\pi/\sqrt{2} \approx 4.443 \), the frequency is \( 1/4.443 \approx 0.225 \), and the phase shift is \( -0.374/\sqrt{2} \approx -0.264 \). A graph of this equation of motion is shown in Figure 3.9. Notice that the oscillations are very hard to see because they are quickly damped out due to the relatively large negative exponent in \( e^{-2t} \).
Figure 3.9: Graph of \( x(t) = 1.367e^{-2t} \sin(\sqrt{2}t + 0.374) \)

**Vertical motion**

The systems considered in this section have been described as objects vibrating horizontally. Now consider an object hanging from a spring as in Figure 3.10.

Figure 3.10: An object vibrating vertically

The object comes to rest at its rest position (also called an *equilibrium point*). If the object is pushed up or down, it begins vibrating. Let \( x(t) \) denote the distance of the object below the rest position (a positive value of \( x(t) \) means the object is below its rest position, a negative value means the opposite). It can be shown that \( x(t) \) is described by the same DE
as in a horizontal motion system,

\[ mx'' + cx' + kx = 0. \]

This DE can also be used to describe the motion of a system much larger than an object on a spring, as illustrated in the next example.

**Example 3.2.4** Suppose the motion of the tip of an airplane wing is modeled as a mass-spring system with mass of 500 kg, damping constant 2000 N·s/m, and spring constant 100000 N/m. Find the equation of motion following a 2 m initial displacement with zero initial velocity, and find the period and phase shift. Also find the first three times at which the wing is at its equilibrium point.

**Solution.** We are given \( m = 500 \), \( k = 100000 \), and \( c = 2000 \), so that DE we want to solve is \( 500x'' + 2000x' + 100000x = 0 \). The characteristic polynomial is \( 500r^2 + 2000r + 100000 \) and its roots are

\[ r = \frac{-2000 \pm \sqrt{2000^2 - 4(500)(100000)}}{2(500)} = \frac{-2000 \pm \sqrt{-1.96 \times 10^8}}{1000} = -2 \pm 14i. \]

Since the discriminate is negative, the system is under-damped. The initial conditions are given as \( x(0) = 2 \) and \( x'(0) = 0 \). Following the same procedures as in Example 3.2.3, we find that the equation of motion is

\[ x(t) = e^{-2t} \left[ 2 \cos(14t) + \frac{2}{7} \sin(14t) \right] \]

(the reader should verify this). The period is then \( 2\pi/14 \approx 0.449 \). To find the phase shift, we again follow the same procedures as in Example 3.2.3 and find that \( A \approx 2.02 \) and \( \phi = -1.429 \) so that the phase shift is \( -1.429/14 \approx -0.102 \). Using these quantities, the equation of motion can be described as

\[ x(t) = 2.02e^{-2t} \sin(14t + 1.429). \]  

(3.5)

To find the first three times at which the wing is at its equilibrium point, we make two observations from Figure 3.7:

1. If the value of the phase shift is denoted \( s \), then \( x(s) = 0 \).
2. If the value of the period is denoted \( p \), then

\[ x(s + 0.5p) = 0, \quad x(s + p) = 0, \quad x(s + 1.5p) = 0, \quad \text{etc.} \]

In words, \( x(t) \) will equal 0 at each half-period after \( x(t) = 0 \).
In this problem, the phase shift is \(-0.102\) and the period is \(0.449\). Thus \(x(-0.102) = 0\) (the reader can easily verify this calculation by substituting \(t = -0.102\) into equation (3.5)) and \(x(t)\) will also equal 0 at

\[
t = -0.102 + 0.5(0.449) = 0.1225, \quad t = -0.102 + (0.449) = 0.347,
\]

and \(t = -0.102 + 1.5(0.449) = 0.5715\).

Figure 3.11 shows a graph of the motion and graphically confirms these calculations. Notice that the displacement asymptotically approaches 0, meaning the wing will eventually stabilize.

**Figure 3.11:** Graph of \(x(t) = 2.02e^{-2t}\sin(14t + 1.429)\)

**Exercises** For exercises 1-6, find a function that describes the displacement of the mass of the given mass-spring system. Sketch a graph of the function on the interval \(0 \leq t \leq 5\). Classify the type of damping as over-damped, under-damped or critically-damped.

3.2.1 Mass 1 kg, no damping, spring constant \(64 \frac{m}{n}\). The mass is displaced 0.3 m downward and released.

3.2.2 Mass 1 kg, no damping, spring constant \(64 \frac{m}{n}\). The mass is in its rest position and hit with a hammer which gives it an initial velocity of \(\frac{15}{4}\) in the upward direction.

3.2.3 Mass 1 kg, damping constant \(12 \frac{n}{m}\), spring constant \(72 \frac{n}{m}\). The mass is displaced 1 m upward and released.

3.2.4 Mass 1 kg, damping constant \(12 \frac{n}{m}\), spring constant \(72 \frac{n}{m}\). The mass is in its rest position and hit with a hammer which gives it an initial velocity of \(2 \frac{m}{t}\) in the upward direction.
3.2.5 Mass 3 kg, damping constant \( \frac{48 \text{Ns}}{m} \), spring constant \( \frac{84 \text{N/m}}{m} \). The mass is displaced 2 m upward and released.

3.2.6 Mass 3 kg, damping constant \( \frac{48 \text{Ns}}{m} \), spring constant \( \frac{84 \text{N/m}}{m} \). The mass is displaced 2 m upward and simultaneously hit with a hammer, imparting a velocity of \( \frac{1 \text{m}}{s} \) in the downward direction.

For exercises 7-10, find a function that describes the displacement of the mass of the given mass-spring system. Provide a table of values in 1 second intervals for \( 0 \leq t \leq 5 \). Classify the type of damping as over-damped, under-damped or critically-damped, and if the motion is periodic, find the period and frequency of the oscillations.

3.2.7 Mass 1 kg, damping constant \( \frac{20 \text{Ns}}{m} \), spring constant \( \frac{100 \text{N/m}}{m} \). The mass is displaced 0.5 m downward and given an initial velocity of \( \frac{2 \text{m}}{s} \) in the downward direction.

3.2.8 Mass 1 kg, damping constant \( \frac{3 \text{Ns}}{m} \), spring constant \( \frac{21 \text{N/m}}{m} \). The mass is displaced 1 m upward and released.

3.2.9 Mass 5 kg, damping constant \( \frac{10 \text{Ns}}{m} \), spring constant \( \frac{1 \text{N/m}}{m} \). The mass is displaced 1 m upward and released with a velocity of \( \frac{1 \text{m}}{s} \) in the downward direction.

3.2.10 Mass 5 kg, damping constant \( \frac{5 \text{Ns}}{m} \), spring constant \( \frac{1 \text{N/m}}{m} \). The mass is displaced 1 m upward and released with a velocity of \( \frac{1 \text{m}}{s} \) in the downward direction.

3.2.11 Show that in an over-damped system, both the roots of the characteristic polynomial are negative. (Hint: Argue that \( c^2 - 4mk < c^2 \)).

3.2.12 Consider an over-damped system whose equation of motion has the form

\[ x(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}. \]

We will show that the function \( x(t) \) has at most one local minimum or maximum value.

a. Consider a function of the form \( y(t) = a + be^{ct} \) where \( a, b, \) and \( c \) are non-zero constants. Show that \( y(t) = 0 \) at \( t = \ln(-a/b)/c \) if and only if \(-a/b > 0 \)

b. Find the derivative of the equation of motion, \( x'(t) \).

c. Assuming that \( C_1 \) and \( C_2 \) are non-zero, use part a. to show that \( x'(t) \) can equal 0 for at most one value of \( t \). (Hint: Set \( x'(t) = 0 \) and multiply both sides by \( e^{-r_1 t} \). Then apply the results of part a.)

3.2.13 Consider an over-damped system whose equation of motion has the form

\[ x(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}. \]

where \( r_1 \) and \( r_2 \) are the distinct real roots of the characteristic polynomial. Assume that \( r_1 < r_2 \).
a. Show that if \( C_1 \) and \( C_2 \) have the same sign, then the object never crosses its rest position.  
(\textbf{Hint}: Exponential functions are always positive.)

b. Show that if \( C_1 \) and \( C_2 \) have opposite signs, and \(-C_2/C_1 < 1\), then the object will cross its rest position exactly once at time
\[
t = \frac{\ln(-C_2/C_1)}{r_1 - r_2}.
\]

3.2.14 Consider a critically-damped system whose equation of motion has the form
\[
x(t) = C_1 e^{rt} + C_2 te^{rt}.
\]

a. Show that if \( C_2 \neq 0 \), then the object crosses its rest position exactly once at time \( t = -C_1/C_2 \).

b. Show that if \( C_2 = 0 \), then the object will never cross its rest position (unless \( C_1 \) also equals 0, in which case the object never moves at all).

3.2.15 A damped mass-spring system as described in this section has mass \( m = 1 \) kg, spring constant \( k = 2 \) newtons/meter and damping constant \( c = 2 \) newtons/m/s. No external force is acting on the mass. Let \( x \) represent the position of the mass in meters and let \( t \) represent time in seconds. If the mass is started in motion by displacing it one meter in the positive \( x \) direction and then released, then the initial conditions would be \( x(0) = 1 \) and \( x'(0) = 0 \) (since \( x' \) represents the velocity and when released the mass has velocity zero).

a. Write the differential equation of motion for the system.

b. Find an explicit solution to the differential equation from part a. subject to the initial conditions \( x(0) = 1 \) and \( x'(0) = 0 \). Graph the solution \( x \) as a function of \( t \) and describe what this graph represents in terms of the mass-spring system.

3.3 Nonhomogeneous Equations and The Method of Undetermined Coefficients

The previous two sections dealt with solving equations where the right-hand sides are 0, called homogeneous equations. We now turn our attention to solving equations where the right-hand sides are not 0, called nonhomogeneous equations. Specifically we solve equations of the form
\[
a x'' + b x' + c x = f(t)
\]  \hspace{1cm} (3.6)
where \( f(t) \neq 0 \) is a function of \( t \). The method for solving such equations that we present in this section applies only to equations where the function \( f(t) \) is of certain forms (to be
described later). Later in this text, we present more general methods for solving nonhomogeneous equations.

The next theorem, which we present without proof, motivates our method for finding the general solution to Equation (3.6).

**Theorem 3.5** If \( x_h(t) \) is the general solution of the homogeneous equation \( ax'' + bx' + cx = 0 \) and \( x_p(t) \) is any solution to Equation (3.6), then the general solution to Equation (3.6) is the sum
\[
x = x_h + x_p.
\]

The solution \( x_p \) to Equation (3.6) is called a *particular solution*. This theorem suggests two general steps for solving a nonhomogeneous equation:

1. Find the general solution to the corresponding homogeneous equation, \( x_h \).
2. Find a particular solution, \( x_p \).

To find \( x_h \) we can use techniques from Section 3.1. The next example illustrates these steps.

**Example 3.3.1** Consider the DE
\[
x'' - 4x' - 12x = 3e^{5t}.
\]
Show that \( x_p(t) = -\frac{3}{t} e^{5t} \) is a particular solution and find the general solution.

**Solution.** To verify the particular solution, we simply need to plug \( x_p \) into the DE and verify that both sides are equal:
\[
\frac{-75}{t} e^{5t} - 4 \left( -\frac{15}{t} e^{5t} \right) - 12 \left( -\frac{3}{t} e^{5t} \right) = 3e^{5t}.
\]
A quick check of the arithmetic on the left-hand side verifies that both sides really are equal. To find the general solution, we need to solve the homogeneous equation
\[
x'' - 4x' - 12x = 0
\]
using the techniques from Section 3.1. The roots of the characteristic polynomial are \(-2\) and \(6\) (the reader should verify this) so that the general solution to the homogeneous equation

\[
x = c_1 e^{-2t} + c_2 e^{6t}. 
\]
is
\[ x_h(t) = C_1 e^{-2t} + C_2 e^{5t}. \]

Therefore, the general solution to the nonhomogeneous equation is
\[ x(t) = C_1 e^{-2t} + C_2 e^{5t} - \frac{3}{7} e^{5t}. \]

Finding \( x_h \) is relatively easy, but how do we find \( x_p \)? The key lies in the following observation: In Example 3.3.1, the right-hand side of the nonhomogeneous equation, \( f(t) \), and \( x_p \) have the same basic form (they are both exponential functions with the same exponents) but have different coefficients. This suggests that we find a function \( x_p \) with the same basic form as \( f(t) \), and then determine the necessary values of the coefficients. This is called the method of undetermined coefficients.

**Method of Undetermined Coefficients**

**Purpose**: To find the general solution to equation (3.6) where the right-hand side, \( f(t) \) is of one of the forms described in Table 3.2.

1. Choose \( x_p \) with the same basic form as \( f(t) \) as described in Table 3.2. If \( f(t) \) is the sum of terms of different forms, the \( x_p \) is the sum of the corresponding forms.
2. Find the general solution to the corresponding homogeneous equation, \( x_h \), using the techniques from Section 3.1.
3. If \( x_p \) and \( x_h \) share any similar terms (meaning terms that only differ by the constant in front), then multiply the shared terms in \( x_p \) by \( t \). Repeat if necessary.
4. Plug \( x_p \) into the DE and find the values of the coefficients.
5. The general solution is \( x = x_h + x_p \).

Note that Table 3.2 describes only a few possible forms of \( f(t) \). Exercises 3.3.15 and 3.3.16 describe a few other forms. In step 3 we might wonder why we would multiply repeated terms by \( t \). To motivate this step, remember that in Section 3.1 when there was a repeated root, the second term in the general solution is the first term multiplied by \( t \).
3.3 Nonhomogeneous Equations and The Method of Undetermined Coefficients

### Table 3.2: Method of Undetermined Coefficients, Step 1

<table>
<thead>
<tr>
<th>Form of $f(t)$</th>
<th>Form of $x_p$</th>
<th>Example $f(t)$</th>
<th>Example $x_p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$ae^{kt}$</td>
<td>$Ae^{kt}$</td>
<td>$3e^{2t}$</td>
<td>$Ae^{2t}$</td>
</tr>
<tr>
<td>$a \cos(\omega t) + b \sin(\omega t)$</td>
<td>$A \cos(\omega t) + B \sin(\omega t)$</td>
<td>$4 \sin(3t)$</td>
<td>$A \cos(3t) + B \sin(3t)$</td>
</tr>
<tr>
<td>$a + bt + ct^2 + \cdots + zt^n$</td>
<td>$A + Bt + Ct^2 + \cdots + Zt^n$</td>
<td>$3 + 2t + 5t^3$</td>
<td>$A + Bt + Ct^2 + Dt^3$</td>
</tr>
</tbody>
</table>

**Example 3.3.2** Find the general solution to the DE $x'' - x' = 6e^{2t}$.

**Solution.** We follow the steps of the method of undetermined coefficients.

1. From the second row of Table 3.2 we assume a particular solution of the form $x_p = Ae^{2t}$.
2. The characteristic polynomial is $r^2 - r$ which has roots of 0 and 1. Thus the general solution to the homogeneous equation is $x_h = C_1 + C_2 e^t$.
3. Note that $x_p$ and $x_h$ do not share any similar terms, so no adjustment to $x_p$ is necessary.
4. Plugging $x_p$ into the DE gives

$$4Ae^{2t} - 2Ae^{2t} = 6e^{2t}.$$ 

Combining terms on the left-hand side gives

$$2Ae^{2t} = 6e^{2t}$$

which means that $A = 3$. Thus a particular solution is $x_p = 3e^{2t}$.
5. The general solution is $x(t) = C_1 + C_2 e^t + 3e^{2t}$.

In Example 3.3.2, determining the value of the coefficient $A$ in step 4 was rather trivial because all the terms in the equation were like terms (they were all of the form $e^{2t}$). In equations where this is not the case, we use the principle that the coefficients of like terms on both sides of the equation must be equal. The next example illustrates this principle.

**Example 3.3.3** Find the general solution to the DE $x'' + 2x' + x = 4 \sin(3t)$ and describe its long-term behavior.

**Solution.** We follow the steps of the method of undetermined coefficients.
1. From the first row of Table 3.2 we assume a particular solution of the form \( x_p = A \cos(3t) + B \sin(3t) \).

2. The characteristic polynomial is \( r^2 + 2r + 1 \) which has a single real root of \(-1\). Thus the general solution to the homogeneous equation is \( x_h = C_1 e^{-t} + C_2 te^{-t} \).

3. Note that \( x_p \) and \( x_h \) do not share any similar terms, so no adjustment to \( x_p \) is necessary.

4. Plugging \( x_p \) into the DE gives

\[
\begin{align*}
-9A \cos(3t) - 9B \sin(3t) + 2(-3A \sin(3t) + 3B \cos(3t)) + A \cos(3t) + B \sin(3t) &= 4 \sin(3t) \\
x_p' &= x_p \\
x_p' &= x_p
\end{align*}
\]

Multiplying this out, and collecting like terms we get

\[
(-8A + 6B) \cos(3t) + (-6A - 8B) \sin(3t) = 0 \cos(3t) + 4 \sin(3t).
\]

We now equate coefficients of like terms on both sides of the equation and get

\[
\begin{align*}
\text{cos}(3t) \text{ terms: } &-8A + 6B = 0 \\
\text{sin}(3t) \text{ terms: } &-6A - 8B = 4
\end{align*}
\]

Solving this system of linear equations (either by hand or with the aid of software) yields

\[
A = -\frac{6}{25} = -0.24, \quad \text{and} \quad B = -\frac{8}{25} = -0.32
\]

so that the a particular solution is

\[
x_p = -0.24 \cos(3t) - 0.32 \sin(3t).
\]

5. The general solution is

\[
x(t) = C_1 e^{-t} + C_2 te^{-t} - 0.24 \cos(3t) - 0.32 \sin(3t).
\]

The long-term behavior of this solution is governed completely by \( x_p \) because \( \lim_{t \to \infty} x_h(t) = 0 \). Thus for sufficiently large \( t \), \( x(t) \) oscillates according to the formula given by \( x_p(t) \), regardless of the values of \( C_1 \) and \( C_2 \). Figure 3.12 shows the graph of \( x(t) \) for the arbitrarily chosen values \( C_1 = C_2 = 1 \). \qed
Initial value problems involving nonhomogeneous equations can be solved in the same basic way as other types of equations. First we find the general solution, then we use the initial conditions to find the values of the arbitrary constants.

Example 3.3.4 Solve the IVP

\[ x'' + 3x' + 2x = 3 + 2t + 5t^3, \quad x(0) = 1, \; x'(0) = -1. \]

Graph the solution and describe its long-term behavior.

**Solution.** We first find the general solution to the DE using the method of undetermined coefficients.

1. From the third line of Table 3.2, we assume a particular solution to the DE of the form \( x_p = A + Bt + Ct^2 + Dt^3 \).
2. The characteristic polynomial is \( r^2 + 3r + 2 \) which roots of \(-1\) and \(-2\). Thus the general solution to the homogeneous equation is \( x_h = C_1 e^{-t} + C_2 e^{-2t} \).
3. Note that \( x_p \) and \( x_h \) do not share any similar terms, so no adjustment to \( x_p \) is necessary.
4. Plugging \( x_p \) into the DE gives

\[
\begin{align*}
x_p'' + \left(3(B + 2Ct + 3Dt^2)\right) + 2\left(A + Bt + Ct^2 + Dt^3\right) & = 3 + 2t + 5t^3. \\
\end{align*}
\]

After collecting like terms we get

\[
(2C + 3B + 2A) + (6D + 6C + 2B)t + (9D + 2C)t^2 + (2D)t^3 = 3 + 2t + 0t^2 + 5t^3.
\]
CHAPTER 3  Second-order Differential Equations

By equating coefficients of like terms, we get the four equations

\[\begin{align*}
2C + 3B + 2A &= 3 \\
6D + 6C + 2B &= 2 \\
9D + 2C &= 0 \\
2D &= 5
\end{align*}\]

Solving this system of linear equations yields

\[\begin{align*}
A &= -\frac{225}{8} = -28.125, & B &= \frac{109}{4} = 27.25, & C &= -\frac{45}{4} = -11.25, & D &= \frac{5}{2} = 2.5
\end{align*}\]

so that the particular solution to the DE is

\[x_p = -28.125 + 27.25t - 11.25t^2 + 2.5t^3.\]

5. The general solution to the DE is

\[x = C_1e^{-t} + C_2e^{-2t} - 28.125 + 27.25t - 11.25t^2 + 2.5t^3.\]

Now to solve the IVP, we use the initial conditions to find the values of \(C_1\) and \(C_2\). The derivative of the general solution is

\[x' = -C_1e^{-t} - 2C_2e^{-2t} + 27.25 - 22.5t + 7.5t^2.\]

The initial conditions \(x(0) = 1\) and \(x'(0) = -1\) yield

\[\begin{align*}
1 &= x(0) = C_1 + C_2 - 28.125 \\
-1 &= x'(0) = -C_1 - 2C_2 + 27.25
\end{align*}\]

or equivalently

\[\begin{align*}
C_1 + C_2 &= 29.125 \\
-C_1 - 2C_2 &= -28.25.
\end{align*}\]

The solution to this system of linear equations is \(C_1 = 30.0, C_2 = -0.875\). Thus the solution to the IVP is

\[x(t) = 30.0e^{-t} - 0.875e^{-2t} - 28.125 + 27.25t - 11.25t^2 + 2.5t^3.\]
The graphs of both the solution to the IVP and the particular solution alone are shown in Figure 3.13.

\[ \text{Figure 3.13: Graphs of } x_p \text{ (dotted) and } x(t) \text{ (solid)} \]

Since \( x_h \) contains only exponential terms with negative exponents, \( \lim_{t \to \infty} x_h(t) = 0 \). Thus for large values of \( t \), \( x(t) \approx x_p(t) \). In Figure 3.13 we see that after about \( t = 3 \), \( x(t) \) and \( x_h(t) \) become almost indistinguishable, and both grow without bound.

In step 3 of the previous examples, no adjustment to \( x_p \) was required. In the next example, an adjustment is necessary.

**Example 3.3.5** Find the general solution to the differential equation \( x'' - x = 4e^t \).

**Solution.** We follow the steps of the method of undetermined coefficients.

1. From the first row of Table 3.2 we assume a particular solution of the form \( x_p = Ae^t \).

2. The characteristic polynomial is \( r^2 - 1 \) which has roots of \( \pm 1 \). Thus the general solution to the homogeneous equation is \( x_h = C_1e^t + C_2e^{-t} \).

3. Note that \( x_p \) and \( x_h \) share terms of the form \( Ce^t \), so we multiply \( x_p \) by \( t \) and get \( x_p = Ae^t \).

4. Plugging \( x_p \) into the DE gives

\[
\frac{d}{dt} \left(2 Ae^t + Ate^t\right) - Ate^t = 4e^t.
\]

\[
(2Ae^t + Ate^t) - (Ate^t) = 4e^t.
\]
Since the $Ate^t$ terms cancel out, we are left with $2Ae^t = 4e^t$ which means that $A = 2$. Thus a particular solution is

$$x_p = 2te^t.$$ 

5. The general solution is

$$x(t) = C_1 e^t + C_2 e^{-t} + 2te^t.$$

It is interesting to note what would happen if we didn’t adjust $x_p$ and simply used $x_p = Ae^t$. Substituting this into the DE would yield

$$\frac{(Ae^t)}{x_p'} - \frac{(Ae^t)}{x_p} = 4e^t.$$

But the left-hand side reduces to 0, resulting in $0 = 4e^t$, which is impossible. A result such as this indicates that we had a bad choice of $x_p$. \hfill\square

The next example illustrates a case where the right-hand side of the DE, $f(t)$, has terms of different types.

**Example 3.3.6** Find the general solution to $x'' + 4x = 5 \cos(2t) + 10e^t$.

**Solution.** We follow the steps of the method of undetermined coefficients.

1. Notice that $f(t)$ has terms corresponding to the first and second rows of Table 3.2. Thus we assume a particular solution of the form $x_p = A \cos(2t) + B \sin(2t) + Ce^t$.

2. The characteristic polynomial is $r^2 + 4$ which has roots of $\pm 2i$. Thus the general solution to the homogeneous equation is $x_h = C_1 \cos(2t) + C_2 \sin(2t)$.

3. Note that $x_p$ and $x_h$ share terms of the form $C \cos(2t)$ and $C \sin(2t)$, so we multiply these terms in $x_p$ by $t$ and get $x_p = At \cos(2t) + Bt \sin(2t) + Ce^t$.

4. Plugging $x_p$ into the left-hand side of the DE gives

$$\left\{4B \cos(2t) - 4A \sin(2t) - 4At \cos(2t) - 4Bt \sin(2t) + Ce^t\right\} + 4\left(At \cos(2t) + Bt \sin(2t) + Ce^t\right).$$

After simplifying, collecting like terms, and setting this left-hand side equal to the right-hand side of the DE, we find that

$$4B \cos(2t) - 4A \sin(2t) + 5Ce^t = 5 \cos(2t) + 0 \sin(2t) + 10e^t$$
(the reader should verify all these derivatives and the algebra). By equating coefficients, we get
\[ B = \frac{5}{4} = 1.25, \quad A = 0, \quad \text{and} \quad C = 2 \]
so that a particular solution is
\[ x_p = 1.25t \sin(2t) + 5e^t. \]

5. The general solution is
\[ x(t) = C_1 \cos(2t) + C_2 \sin(2t) + 1.25t \sin(2t) + 2e^t. \]

**Exercises**

**Directions:** Find the general solutions to the following DE’s.

3.3.1 \( y'' - y = 4 \cos(2t) \)

3.3.2 \( y'' + 4y' + 8y = 3e^{2t} \)

3.3.3 \( y'' + y = 4 \cos(t) \)

3.3.4 \( y'' + 4y' + 4y = 4e^{-2t} \)

3.3.5 \( y'' - 2y' + y = 2t \)

3.3.6 \( y'' + 5y' + 6y = 2e^{4t} \)

3.3.7 \( y'' - 3y' + 2y = e^t \)

3.3.8 \( y'' + 16y = \sin(3t) \)

3.3.9 \( y'' + 10y' + 21y = e^t + t^2 \)

3.3.10 \( y'' + 9y = \sin(t) - e^{2t} \)

**Directions:** Solve the following IVP’s.

3.3.11 \( y'' + 2y' + 2y = 2 \cos(t), \quad y(0) = 1, \quad y'(0) = -1 \)

3.3.12 \( y'' + y = -\sin(t), \quad y(0) = 0, \quad y'(0) = 0 \)

3.3.13 \( y'' - 9y = t - \cos(t), \quad y(0) = 0, \quad y'(0) = 1 \)
3.3.14 $y'' + 2y' + y = \sin(t) + \cos(2t)$, $y(0) = 0$, $y'(0) = 0$

3.3.15 If $f(t)$ has terms of the form $e^{at} \cos(\omega t)$, or $e^{at} \sin(\omega t)$, we can assume a particular solution of the form $x_p = A e^{at} \cos(\omega t) + B e^{at} \sin(\omega t)$. Then proceed with the remaining steps in the method of undetermined coefficients. Use this approach in the following problems.

a. Find the general solution to $x'' + x = e^{-t} \cos(t)$.

b. Find the solution to the IVP $x'' + 2x' + 2x = e^{-t} \sin(t)$, $x(0) = 0$, $x'(0) = 0$

3.3.16 If $f(t)$ has terms of the form $te^{bt}$, $t \cos(\omega t)$, or $t \sin(\omega t)$, try the following to find $x_p$: Replace the coefficients $A$ and $B$ in the second column of Table 3.2 with coefficients of the form $(A_1 + A_2 t)$ and $(B_1 + B_2 t)$. Then proceed with the remaining steps in the method of undetermined coefficients. Use this approach in the following problems.

a. Find the general solution to $x'' + x = 3t \sin(2t)$.

b. Find the solution to the IVP $x'' + 2x' + x = te^t$, $x(0) = 1$, $x'(0) = 0$.

3.3.17 Consider the nonhomogeneous DE $ax'' + bx' + cx = f(t)$ where $f(t) \neq 0$. Show that if $x_p$ is a particular solution to the equation, then $x_p$ cannot be a solution to the corresponding homogeneous equation $ax'' + bx' + cx = 0$.

3.3.18 In Example 3.3.5, suppose we did not adjust $x_p$ and used $x_p = Ae^t$. Try plugging this $x_p$ into the DE and solving for $A$. What do you get? What does this mean about the possibility of particular solutions of the form $x_p = Ae^t$?

3.4 Driven Mass-Spring Systems, Beats, and Resonance

Consider a damped mass-spring system as described in Section 3.2, but with a new twist. Our system will now have an external driving force. If you have ever observed a washing machine wobbling as it spins (often due to an unbalanced arrangement of clothes inside the machine) then you have experienced a periodic driving force. Another example of the same phenomenon is an airplane engine with a broken propeller. When placed on an airplane wing, the rotational motion of the propeller is transformed into a periodic, or sinusoidal, driving force causing the wing to vibrate up and down. In fact, since no real propeller system can be perfectly balanced, this is something that engineers must consider.

We will model the system in the same way as in Section 3.2, but with the addition of an external force described by a function $f(t)$ as illustrated in Figure 3.14. Such a system is called a driven mass-spring system. We can think of the object moving horizontally, as shown in the figure, or vertically as if it were hanging on the spring. As shown in the introduction to this chapter, the DE describing $x(t)$ is

$$mx'' + cx' + kx = f(t).$$

(3.7)
When \( f(t) \neq 0 \), this equation is called the *driven mass-spring equation*. This equation is a non-homogeneous, constant coefficient, linear, second-order differential equation, just the type we studied in Section 3.3. Since we already know how to solve such equations, in this section we concentrate on the interpretation of the solution. The point of view that we will take is that \( f(t) \) is the *input* to the system, and the solution to the DE \( x(t) \) is the *response* or *output*.

As shown in Section 3.3, the general solution to an equation such as (3.7) has the form

\[
x = x_h + x_p
\]

where \( x_p \) is a particular solution to the equation and \( x_h \) is the general solution to the homogeneous equation \( mx'' + cx' + kx = 0 \). In terms of a forced mass-spring system, we call \( x_h \) the *transient* part of the response (at least when damping is present), and \( x_p \) the *steady-state* part of the response. The next example illustrates the meanings of these names.

**Example 3.4.1** A 2 kg object is attached to a spring with spring constant 30 N/m and damping constant 16 N·m. The object is neither displaced nor given an initial velocity. Find the response of the system if the input is a sinusoidal driving force with amplitude 400 newtons and frequency of \( 6/\pi \) Hz beginning its cycle at \( t = 0 \). Graph the response function and relate the behavior to the input driving function.

**Solution:** We are given \( m = 2 \), \( c = 16 \), and \( k = 30 \). The “sinusoidal driving force” means that the external force is described by a function of the form \( f(t) = A \sin(\omega t) \) where \( A \) is the amplitude and the frequency is \( \omega/(2\pi) \). We are given \( A = 400 \) and a frequency of \( 6/\pi \),
so
\[
\frac{6}{\pi} = \frac{\omega}{2\pi} \implies \omega = 12.
\]
Thus \( f(t) = 400 \sin(12t) \), and the DE we want to solve is
\[
2x'' + 16x' + 30x = 400 \sin(12t).
\]
Because the object is neither displaced nor given an initial push, we have the initial conditions \( x(0) = 0 \) and \( x'(0) = 0 \). To solve this IVP, we first find the general solution to the DE using the method of undetermined coefficients (we omit most of the details). A particular solution has the form
\[
x_p = A \cos(12t) + B \sin(12t).
\]
The general solution to the corresponding homogeneous equation is
\[
x_h = C_1 e^{5t} + C_2 e^{-3t}.
\]
Note that \( x_h \) shares no terms with \( x_p \) so no adjustment to \( x_p \) is necessary. The values of the coefficients \( A \) and \( B \) are
\[
A = -\frac{6400}{8619} \approx -0.74255 \quad \text{and} \quad B = -\frac{8600}{8619} \approx -0.99780.
\]
The general solution to the DE is then
\[
x = C_1 e^{5t} + C_2 e^{-3t} - 0.74255 \cos(12t) - 0.99780 \sin(12t).
\]
The initial conditions yield \( C_1 = -7.1008 \) and \( C_2 = 7.8434 \) so that the solution to the IVP is
\[
x(t) = -7.1008 e^{5t} + 7.8434 e^{-3t} - 0.74255 \cos(12t) - 0.99780 \sin(12t).
\]
A graph of this function is shown in Figure 3.15. From the figure we see that after roughly \( t = 2 \), the graph has “settled down” into a steady state, a simple sinusoidal oscillation corresponding to the particular, or steady state, part of the response, \( x_p = -0.74255 \cos(12t) - 0.99780 \sin(12t) \). This is because the homogeneous, or transient, part of the response, \( x_h = -7.1008 e^{-5t} + 7.8434 e^{-3t} \), “dies out” due to the negative exponents.

To compare the steady state part of the response, \( x_p \), to the driving function \( f(t) = 400 \sin(12t) \), we can use ideas from Theorem 3.4 to write \( x_p \) in the form \( x_p = D \sin(12t - \phi) \). To this end, we have
\[
D = \sqrt{(-0.74255)^2 + 0.99780^2} \approx 1.2438
\]
For $\phi$ we first try $\arctan(-(-0.74255)/(-0.99780)) \approx -0.63977$, but this value does not work since

$$-D \sin(-0.63977) = -1.2438 \sin(-0.63977) \approx 0.74256$$

which does not equal $-0.74255$ (even within rounding error). Thus we use

$$\phi = -0.63977 + \pi \approx 2.5018,$$

and we can write

$$x_p = 1.2438 \sin(12t - 2.5018).$$

Comparing this form of $x_p$ to $f(t)$, we see that the graphs of $x_p$ and $f(t)$ will look very similar, but $x_p$ is shifted to the right $2.5018/12 \approx 0.20848 \, \text{s}$. Since the driving function oscillates at $6/\pi$ Hz, we can say that $x_p$ is shifted to the right

$$\left( \frac{6 \text{ cycles}}{\pi} \right) (0.20848 \, \text{s}) \approx 0.39817 \text{ cycles.}$$

Thus we could say that in the long-term, the response lags behind the driving force by about $0.4$ cycles. The driving function and steady state response are plotted in Figure 3.16 to illustrate this lag. Note the different vertical axes for the different functions.
Beats

A phenomenon called \textit{beats} is often observed in music and engineering. To motivate this idea, explain where it comes from, and define some related terms, consider the following example involving an undamped forced mass-spring system.

\textbf{Example 3.4.2} A 1 kg object is attached to a spring with spring constant 4 N/m and no damping. The object is neither displaced nor given an initial velocity. Find the response of the system if the input is a sinusoidal driving force with amplitude 1 newton and frequency of $1.7/(2\pi)$ Hz, beginning its cycle at \(t = 0\). Graph the response function and discuss the behavior.

\textbf{Solution.} We are given \(m = 1\), \(c = 0\), and \(k = 4\). The driving function is of the form \(f(t) = A\sin(\omega t)\) where \(A = 1\) and

\[
\frac{1.7}{2\pi} = \frac{\omega}{2\pi} \implies \omega = 1.7.
\]

Thus the differential equation we need to solve is \(x'' + 4x = \sin(1.7t)\), and the initial conditions are \(x(0) = 0\) and \(x'(0) = 0\). We solve this IVP using the method of undetermined coefficients. Again, we omit most of the details. A particular solution has the form

\[x_p = A\cos(1.7t) + B\sin(1.7t).\]

The general solution to the corresponding homogeneous equation is

\[x_h = C_1\cos(2t) + C_2\sin(2t).\]
Note that $x_h$ describes the response if there were no external force. Stated another way, $x_h$ describes the motion if the object were allowed to vibrate naturally. The motion is periodic with period $2\pi/2 = \pi$ and frequency $2/(2\pi) = 1/\pi$. We call this frequency the natural frequency. The driving function $f(t) = \sin(1.7t)$ has a frequency of $1.7/(2\pi)$. We call this frequency the driving frequency.

The values of the coefficients $A$ and $B$ are $A = 0$ and $B = 1/1.11 \approx 0.9009$, so that a particular solution is

$$x_p = 0.9009 \sin(1.7t)$$

and the general solution is

$$x(t) = C_1 \cos(2t) + C_2 \sin(2t) + 0.9009 \sin(1.7t).$$

The initial conditions $x(0) = 0$ and $x'(0) = 0$ yield $C_1 = 0$ and $C_2 \approx -0.7658$ so that the solution to the IVP is

$$x(t) = -0.7658 \sin(2t) + 0.9009 \sin(1.7t). \quad (3.8)$$

The graph of this solution is shown in Figure 3.17. Notice that the amplitude of the oscillations is rather small near $t = 0$, then the amplitude gets large near $t = 10$, then small near $t = 20$, and so on. This behavior is known as beats.

![Figure 3.17: Graph of $x(t) = -0.7658 \sin(2t) + 0.9009 \sin(1.7t)$](image)

To explain why the system in Example 3.4.2 experiences beats, notice that the coefficients in $x(t)$ are close to $-1$ and $1$, respectively. This means $x(t) \approx -\sin(2t) + \sin(1.7t)$. We can
rewrite this approximation using the trigonometric identity

$$\sin(c) - \sin(d) = 2 \sin\left(\frac{c - d}{2}\right) \cos\left(\frac{c + d}{2}\right)$$

(see Exercise 3.4.9) as

$$x(t) \approx \sin(1.7t) - \sin(2t) = 2 \sin\left(\frac{1.7t - 2t}{2}\right) \cos\left(\frac{1.7t + 2t}{2}\right) = 2 \sin(-0.15t) \cos(1.85t).$$

The cos term in this expression oscillates with a frequency of $1.85/(2\pi) \approx 0.294$ Hz while the sin term oscillates much slower with a frequency of $0.15/(2\pi) \approx 0.0239$ Hz. Thus we can think of $x(t)$ as a cosine function with a variable amplitude described by $2 \sin(-0.15t)$. The function $2 \sin(-0.15t)$ is called the envelope function. Figure 3.18 illustrates this idea.

**Figure 3.18:** Graphs of $x = 2 \sin(0.15t)$ (dotted) and $x = 2 \sin(-0.15t) \cos(1.85t)$ (solid)

The frequency of the envelope function, $0.15/(2\pi) \approx 0.0239$ Hz, is called the beat frequency, and the corresponding period, $(2\pi)/0.15 \approx 41.89$ s, is called the beat period. In practical terms, this means that the system in Example 3.4.2 should experience a beat approximately every $42/2 = 21$ seconds. By examining Figure 3.17, we see that this is indeed the case.

If we replace the number 2 with the parameter $\omega_1$ and the number 1.7 with $\omega_2$ in the above derivation, we can generalize the important points of this discussion as follows:

- The system wants to vibrate naturally with a frequency of $\omega_1/(2\pi)$, called the natural frequency.
- The driving force oscillates with a frequency of $\omega_2/(2\pi)$, called the driving frequency.
• \( \omega_1 \) and \( \omega_2 \) are “close” in value, but not equal.

• The system experiences beats with a frequency of approximately \( \frac{\omega_1 - \omega_2}{4\pi} \) and period of approximately \( \frac{4\pi}{\omega_1 - \omega_2} \), called the beat frequency and beat period, respectively.

Our analysis was based in part on the fact that the coefficients of the sine terms in equation (3.8) were close in absolute value. In general, it can be shown that the closer \( \omega_1 \) and \( \omega_2 \) get to each other, the closer in absolute value the coefficients of the sine terms in the equation of motion get (see Exercise 3.4.7).

We motivated the idea of beats in the context of an undamped system with a driving force described by a sine function. These ideas also apply to systems with a small amount of damping or a driving force described by a cosine function (see Exercise 3.4.8). The next example illustrates an application of these ideas to music.

**Example 3.4.3** Two piano strings that are supposed to vibrate at a frequency of 17 Hz are struck at the same time. One of the strings is in tune, but the other vibrates at a frequency of 16.35 Hz. Approximate the beat frequency and beat period of the resulting sound.

**Solution.** We can think of the in-tune string as a mass-spring system that wants to vibrate naturally with a frequency of 16.35 Hz, and the vibrations from the out-of-tune string as an external force that oscillates with a frequency of 17 Hz. The resulting sound is the response of the system. Thus we have

\[
17 = \frac{\omega_1}{2\pi} \quad \Rightarrow \quad \omega_1 \approx 106.81 \quad \text{and} \quad 16.35 = \frac{\omega_2}{2\pi} \quad \Rightarrow \quad \omega_2 \approx 102.73.
\]

The beat frequency and period are approximately

\[
\frac{106.81 - 102.73}{4\pi} \approx 0.32 \text{ Hz} \quad \text{and} \quad \frac{4\pi}{106.81 - 102.73} \approx 3.08 \text{ s}.
\]

A trained piano tuner can hear these beats, which signals that the strings are out of tune, and adjust them accordingly.

**Resonance**

In the discussion of beats, \( \omega_1 \) and \( \omega_2 \) were close in value, but not equal. Notice that as \( \omega_1 \) and \( \omega_2 \) get “closer” in value, meaning \( (\omega_1 - \omega_2) \to 0 \), we have

\[
\text{Beat frequency} = \frac{\omega_1 - \omega_2}{4\pi} \to 0 \quad \text{and} \quad \text{Beat period} = \frac{4\pi}{\omega_1 - \omega_2} \to \infty.
\]
In practical terms, this means that as the natural frequency and the driving frequency get closer in value, the response of the system experiences beats that occur very infrequently, and a long time apart. It can also be shown that the maximum amplitude of these beats gets larger (see Exercise 3.4.7). But what if \( \omega_1 = \omega_2 \)? Informally, our analysis shows that the response of the system will be one long “beat” which continues to rise forever. This is called pure resonance. The next example formalizes this analysis.

**Example 3.4.4** A 1 kg object is attached to a spring with spring constant 4 N/m. It is driven by a sinusoidal driving force with amplitude 1 N and frequency of \( \frac{2}{(2\pi)} \) Hz, beginning its cycle at \( t = 0 \). If there is no initial displacement or initial velocity, find the response of the system and sketch its graph. Discuss the implications of this result.

**Solution:** We need to solve the IVP \( x'' + 4x = \sin(2t) \), \( x(0) = x'(0) = 0 \) using the method of undetermined coefficients. We assume a particular solution of the form

\[
x_p = A \cos(2t) + B \sin(2t).
\]

The general solution to the corresponding homogeneous equation is

\[
x_h = C_1 \cos(2t) + C_2 \sin(2t).
\]

Notice that the natural frequency of the system is \( \frac{2}{(2\pi)} \), the same frequency as the driving force. This is the mathematical cause of resonance. In terms of the solution to the DE, \( x_p \) and \( x_h \) share \( \cos \) and \( \sin \) terms, so we adjust \( x_p \) by multiplying each term by \( t \). Thus we use a particular solution of the form

\[
x_p = A t \cos(2t) + B t \sin(2t).
\]

Plugging \( x_p \) into the DE to solve for \( A \) and \( B \), we get

\[
4A \sin(2t) - 4At \cos(2t) + 4B \cos(2t) - 4Bt \sin(2t) + 4(A t \cos(2t) + B t \sin(2t)) = \sin(2t).
\]

Notice that the \( t \cos(2t) \) and \( t \sin(2t) \) terms cancel out. Equating the coefficients of the \( \cos(2t) \) and \( \sin(2t) \) terms yields \( A = \frac{1}{4} \) and \( B = 0 \). Hence the general solution to the DE is

\[
x(t) = C_1 \cos(2t) + C_2 \sin(2t) - \frac{1}{4} t \cos(2t).
\]

The initial conditions yield \( C_1 = 0 \) and \( C_2 = \frac{1}{8} \) so that the solution to the IVP is

\[
x(t) = \frac{1}{8} \sin(2t) - \frac{1}{4} t \cos(2t).
\]
Figure 3.19 shows a graph of this response. We see that the response is an oscillation with a greater and greater amplitude due to the $t$ in the coefficient of the $\cos$ term. This type of response is what we call resonance. In a real mass-spring system, this ever-increasing amplitude would probably cause the spring to eventually break and the system to be destroyed. At the very least, the spring would get “bent out of shape.”

![Graph of response](image)

**Figure 3.19:** Graphs of $x(t) = \frac{1}{8} \sin(2t) - \frac{1}{4} t \cos(2t)$

Pure resonance occurs in the context of an undamped system. Resonance can also be defined for a system in which there is a “small” amount of damping. This will be discussed more in Chapter 6.

**Exercises**

**Directions:** For each spring-mass system, find the response and sketch its graph over an appropriate interval. If the motion is damped, identify the transient and steady state parts of the motion, and the approximate time when the transient part dies out. If the motion is undamped, state whether beats or resonance are present or not.

3.4.1 $m = 1$, $c = 3$, $k = 2$, driving function $f(t) = 2 \cos(3t)$, initial conditions $y(0) = 1$, $y'(0) = 0$.

3.4.2 $m = 1$, $c = 2$, $k = 2$, driving function $f(t) = 3 \cos(t)$, initial conditions $y(0) = 1$, $y'(0) = 0$.

3.4.3 $m = 1$, $c = 2$, $k = 1$, driving function $f(t) = - \sin(2t)$, initial conditions $y(0) = 0$, $y'(0) = -1$.

3.4.4 $m = 1$, $c = 0$, $k = 4$, driving function $f(t) = -2 \sin(t)$, initial conditions $y(0) = 0$, $y'(0) = 0$.

3.4.5 $m = 1$, $c = 0$, $k = 4$, driving function $f(t) = 2 \cos(2t)$, initial conditions $y(0) = 0$, $y'(0) = 0$. 
3.4.6 $m = 1$, $c = 0$, $k = 100$, driving function $f(t) = 40 \cos(9t)$, initial conditions $y(0) = 0$, $y'(0) = 0$.

3.4.7 Consider an undamped forced system described by the IVP

$$mx'' + x = F_0 \sin(\omega_2 t), \quad x(0) = x'(0) = 0$$

where $m > 0$ is the mass of the object, $F_0 > 0$ is a constant, and $\omega_2 \neq 1/\sqrt{m}$.

a. Show that the solution to this IVP is

$$x(t) = -A \frac{\omega_2}{\omega_1} \sin(\omega_1 t) + A \sin(\omega_2 t), \quad \text{where} \quad A = \frac{F_0}{1 - (\omega_2/\omega_1)^2}$$

and $\omega_1 = 1/\sqrt{m}$.

b. If $\omega_2$ gets closer to $\omega_1$, what happens to the coefficients of the sine terms in $x(t)$ (i.e. do they get closer in absolute value or further apart)?

c. Show that if $\omega_1 \approx \omega_2$, then

$$x(t) \approx 2A \sin\left(\frac{\omega_2 - \omega_1}{2} t\right) \cos\left(\frac{\omega_2 + \omega_1}{2} t\right).$$

d. If $\omega_2$ gets closer to $\omega_1$, what happens to the value of $A$ (i.e. does $A$ get larger in value or closer to 0)? What does this mean about the maximum amplitude of the $x(t)$?

3.4.8 Consider an undamped forced system described by the IVP

$$mx'' + x = F_0 \cos(\omega_2 t), \quad x(0) = x'(0) = 0$$

where $m > 0$ is the mass of the object, $F_0 > 0$ is a constant, and $\omega_2 \neq 1/\sqrt{m}$. Use the trigonometric identity

$$\cos(c) - \cos(d) = -2 \sin\left(\frac{c + d}{2}\right) \sin\left(\frac{c - d}{2}\right)$$

to show that the solution to this IVP is

$$x(t) = -2B \sin\left(\frac{\omega_2 + \omega_1}{2} t\right) \sin\left(\frac{\omega_2 - \omega_1}{2} t\right), \quad \text{where} \quad B = \frac{F_0}{1 - (\omega_2/\omega_1)^2}$$

and $\omega_1 = 1/\sqrt{m}$.

3.4.9 Prove the trigonometric identity

$$\sin(c) - \sin(d) = 2 \cos\left(\frac{c + d}{2}\right) \sin\left(\frac{c - d}{2}\right)$$

by following these steps:
a. Start with the well-known identity \( \sin(a + b) = \cos a \sin b + \cos b \sin a \) and replace \( a + b \) with \( a - b = a + (-b) \) to get an identity for \( \sin(a - b) \). Use the facts that \( \sin(-x) = -\sin(x) \) and \( \cos(-x) = \cos(x) \) to simplify this identity as much as possible.

b. Subtract these two identities to get an identity for \( \sin(a + b) - \sin(a - b) \).

c. Let \( c = a + b \) and \( d = a - b \). Solve for \( a \) and \( b \) in terms of \( c \) and \( d \).

d. Replace all the \( a \)'s and \( b \)'s with the \( c \)'s and \( d \)'s in the identity for \( \sin(a + b) - \sin(a - b) \) and simplify as much as possible.

Fill in the details and compose a well-written proof.
3.5 Numerical Methods for Second-Order DE’s and Systems

The methods we have developed for finding explicit solutions to second-order DE’s are quite limited in that they apply only to DE’s of a certain type. Different types of equations must be solved with different, and typically more complex, methods, or their solutions must be approximated with numeric methods. In this section we describe some basic techniques for numerically approximating solutions to second-order IVP’s. These techniques are very similar to those presented in Section 2.4.

Numerical techniques have been programmed in most computer algebra systems, such as Maple, Mathematica, and Matlab, and are also available on advanced calculators. It is important to have some idea of how these techniques work, so that we can better understand what the graphs and numbers the software produce mean. Sometimes numerical approximations are quite good, and sometimes they are very poor (as we have already seen with first-order equations). Understanding how numerical methods work help us distinguish the good ones from the poor ones.

The IVP’s we deal with in this section have the general form

\[ x'' = f(t, x, x') , \quad x(t_0) = x_0 , \quad x'(t_0) = y_0 \]  \hspace{1cm} (3.9)

where \( t_0, x_0, \) and \( y_0 \) are given numbers. The right-hand side of the DE, \( f(t, x, x') \), is a function of the independent variable \( t \), the dependent variable \( x \), and the derivative of the dependent variable \( x' \). As with all other general forms of DE’s, the independent variable does not have to be \( t \) and the dependent variable does not have to be \( x \).

Converting Second-Order Equations to Systems

Our approach to implementing numerical methods for second-order (and other higher-order) IVP’s is to first convert the DE into a system of two first-order DE’s, and then use algorithms very similar to those in Section 2.4 to approximate the solution to the first-order equations. To convert the DE in (3.9) to a system of first-order equations, we define a new function \( y \) as \( y = x' \). Then

\[ y' = x'' = f(t, x, x') = f(t, x, y) . \]
Therefore, a system with initial conditions that is equivalent to the IVP (3.9) is

\[
\begin{align*}
x' &= y \\
y' &= f(t, x, y) \\
x(t_0) &= x_0, \quad y(t_0) &= y_0.
\end{align*}
\tag{3.10}
\]

**Example 3.5.1** Convert the IVP

\[
x'' = 4e^x \cos t - 6x^2 + 8(x')^3, \quad x(2) = 3, \quad x'(2) = -1.5
\]

into an equivalent IVP involving a system of first-order equations.

**Solution.** Notice that this IVP involves a non-linear, non-homogeneous DE, a type that would be extremely difficult, if not impossible, to solve analytically. We let \( y = x' \), and have

\[
\begin{align*}
x' &= y \\
y' &= 4e^x \cos t - 6x^2 + 8(x')^3 \\
x(2) &= 3, \quad y(2) &= -1.5.
\end{align*}
\]

\[\Box\]

**Euler’s Method for Systems**

As described in Section 1.2, Euler’s method is a technique for approximating the solution to a first-order IVP of the form

\[
y' = f(t, y), \quad y(t_0) = y_0.
\tag{3.11}
\]

If \( \Delta t \) is a small positive number, called the *step size*, and \( n \) is a positive integer, then with Euler’s method we calculate the points \((t_1, y_1), \ldots, (t_n, y_n)\) using the formulas

\[
\begin{align*}
t_{i+1} &= t_i + \Delta t \\
y_{i+1} &= y_i + f(t_i, y_i) \Delta t
\end{align*}
\]
for $i = 0, \ldots, n - 1$. Then $y_i \approx y(t_i)$ for each $i$. Now consider an IVP involving a system of first-order DE’s of the form

\[
\begin{align*}
    x' &= g(t, x, y) \\
y' &= f(t, x, y) \\
x(t_0) &= x_0, & y(t_0) &= y_0
\end{align*}
\]

(3.12)

where both $x$ and $y$ are functions of the independent variable $t$. Notice that the system in (3.12) consists of two IVP’s of the same basic form as (3.11). This suggests we can use a two-dimensional extension of Euler’s method to solve the system (3.12):

\[
\begin{align*}
t_{i+1} &= t_i + \Delta t \\
x_{i+1} &= x_i + g(t_i, x_i, y_i) \Delta t \\
y_{i+1} &= y_i + f(t_i, x_i, y_i) \Delta t.
\end{align*}
\]

Then $x_i \approx x(t_i)$ and $y_i \approx y(t_i)$ for each $i$. The system (3.10) is a special case of system (3.12) where $g(t, x, y) = y$. The next example illustrates an application of this idea.

**Example 3.5.2** A 1 kg object is attached to a spring with spring constant 1 N/m and no damping. The object is displaced 1 m to the right and released. Estimate the position of the object after 2 s using Euler’s method with 10 steps. Also find the exact position after 2 s, and the error in using Euler’s method. Sketch the exact solution along with the Euler estimates over the interval $0 \leq t \leq 2$.

**Solution:** Using the ideas from Section 3.2, we see that we need to solve the IVP $x'' + x = 0$, $x(0) = 1$, $x'(0) = 0$. This DE is equivalent to $x'' = -x$. To estimate $x(2)$ using Euler’s method, we convert this problem into a system of first-order equations by letting $y = x'$. Then we have

\[
\begin{align*}
x' &= y \\
y' &= -x \\
x(0) &= 1, & y(0) &= 0.
\end{align*}
\]

The formulas for Euler’s method in this problem are

\[
\begin{align*}
t_{i+1} &= t_i + \Delta t \\
x_{i+1} &= x_i + y_i \Delta t \\
y_{i+1} &= y_i - x_i \Delta t.
\end{align*}
\]

Since we want to get to $t = 2$ in 10 steps, we will use steps of size $\Delta t = 2/10 = 0.2$. We
are given \( t_0 = 0 \), \( x_0 = 1 \), and \( y_0 = 0 \), so the first step of Euler’s method is

\[
\begin{align*}
t_1 &= t_0 + \Delta t = 0 + 0.2 = 0.2 \\
x_1 &= x_0 + y_0 \Delta t = 1 + (0)(0.2) = 1.0 \\
y_1 &= y_0 - x_0 \Delta t = 0 - (1)(0.2) = -0.2
\end{align*}
\]

and the second step is

\[
\begin{align*}
t_2 &= t_1 + \Delta t = 0.2 + 0.2 = 0.4 \\
x_2 &= x_1 + y_1 \Delta t = 1.0 + (-0.2)(0.2) = 0.96 \\
y_2 &= y_1 - x_1 \Delta t = -0.2 - (1.0)(0.2) = -0.4
\end{align*}
\]

We continue this for a total of \( n = 10 \) steps. The results are summarized in Table 3.3. From the table, we see that \( x(2) \approx -0.47732 \). Note that the calculations involved with Euler’s method are very tedious to do by hand, so they are almost always performed with software.

<table>
<thead>
<tr>
<th>( i )</th>
<th>( t_i )</th>
<th>( x_i )</th>
<th>( y_i )</th>
<th>( x_i + y_i \Delta t )</th>
<th>( y_i - x_i \Delta t )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>-0.2</td>
</tr>
<tr>
<td>1</td>
<td>0.2</td>
<td>1</td>
<td>-0.2</td>
<td>0.96</td>
<td>-0.4</td>
</tr>
<tr>
<td>2</td>
<td>0.4</td>
<td>0.96</td>
<td>-0.4</td>
<td>0.88</td>
<td>-0.592</td>
</tr>
<tr>
<td>3</td>
<td>0.6</td>
<td>0.88</td>
<td>-0.592</td>
<td>0.7616</td>
<td>-0.768</td>
</tr>
<tr>
<td>4</td>
<td>0.8</td>
<td>0.7616</td>
<td>-0.768</td>
<td>0.608</td>
<td>-0.92032</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>0.608</td>
<td>-0.92032</td>
<td>0.423936</td>
<td>-1.04192</td>
</tr>
<tr>
<td>6</td>
<td>1.2</td>
<td>0.423936</td>
<td>-1.04192</td>
<td>0.215552</td>
<td>-1.12671</td>
</tr>
<tr>
<td>7</td>
<td>1.4</td>
<td>0.215552</td>
<td>-1.12671</td>
<td>-0.00979</td>
<td>-1.16982</td>
</tr>
<tr>
<td>8</td>
<td>1.6</td>
<td>-0.00979</td>
<td>-1.16982</td>
<td>-0.24375</td>
<td>-1.16786</td>
</tr>
<tr>
<td>9</td>
<td>1.8</td>
<td>-0.24375</td>
<td>-1.16786</td>
<td>-0.47732</td>
<td>-1.11911</td>
</tr>
<tr>
<td>10</td>
<td>2</td>
<td>-0.47732</td>
<td>-1.11911</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Table 3.3:** Euler’s Method for a System of two Equations

To find the exact value of \( x(2) \), we can solve IVP using the methods of Section 3.1. The DE \( x'' + x = 0 \) is homogeneous and the characteristic polynomial is \( r^2 + 1 \), the roots of which are \( \pm i \). Thus the general solution is

\[
x(t) = C_1 \cos(t) + C_2 \sin(t).
\]
The initial conditions \( x(0) = 1 \) and \( x'(0) = 0 \) yield \( C_1 = 1 \) and \( C_2 = 0 \) (the reader should verify this). Therefore, the solution to the IVP is \( x(t) = \cos(t) \) and the exact value of \( x(2) \) is \( \cos(2) \approx -0.41615 \).

The error in using Euler’s method to approximate \( x(2) \) is

\[
-0.47732 - (-0.41615) = -0.06117 \text{ m}.
\]

The negative error means that Euler’s method estimates the object is 0.06117 meters to the left of where the object actually is. A graph of the exact solution \( x(t) = \cos(t) \) along with the Euler estimates is given in Figure 3.20. The applet corresponding to this example at uhaweb.hartford.edu/rdecker/DeckerDEbook/DeckerDEbook.html can be used to create similar graphs and tables and can be experimented with interactively.

![Figure 3.20: Euler estimates (boxes) and exact solution \( x = \cos(t) \)](image)

### Order of Convergence and Digits of Accuracy

In section 2.4 we defined a numerical method to be \( k \)th order convergent if dividing the step size by a number \( M \) results in dividing the error by approximately \( M^k \), when \( \Delta t \) is sufficiently small. In that section we showed that Euler’s method for a first-order IVP is of order \( k = 1 \). The same is true of Euler’s method for a system of two equations.

In terms of digits of accuracy, order \( k = 1 \) means that dividing \( \Delta t \) by 10 will result in an error that is divided by approximately \( 10^1 = 10 \). In practical terms, dividing the error
by 10 means we get one more digit of accuracy in our approximation. The next example illustrates this.

**Example 3.5.3** Consider the mass-spring system in Example 3.5.2. Use Euler’s method on available software to estimate \( x(2) \) step sizes of \( \Delta t = 0.2, 0.02, 0.002, \) and \( 0.0002 \). For each case calculate the error involved using Euler’s method and discuss the results in terms of order of convergence.

**Solution.** We use the applet at [http://uhaweb.hartford.edu/rdecker/mathlettoolkit/systemsbook.htm](http://uhaweb.hartford.edu/rdecker/mathlettoolkit/systemsbook.htm) to perform the calculations. To implement Euler’s method and see the numerical results, we select **Euler** in the drop-down box next to **Method** and click on **Display values**. The results are summarized in Table 3.4.

<table>
<thead>
<tr>
<th>Step Size ( \Delta t )</th>
<th>Number of Steps ( n )</th>
<th>Euler Estimate ( x(2) )</th>
<th>Exact Value ( \cos(2) )</th>
<th>Error ( x(2) - \cos(2) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>10</td>
<td>-0.477324902</td>
<td>-0.41614684</td>
<td>-0.0611780659</td>
</tr>
<tr>
<td>0.02</td>
<td>100</td>
<td>-0.42430453</td>
<td>-0.41614684</td>
<td>-0.0081576935</td>
</tr>
<tr>
<td>0.002</td>
<td>1000</td>
<td>-0.416977532</td>
<td>-0.41614684</td>
<td>-0.0008306952</td>
</tr>
<tr>
<td>0.0002</td>
<td>10000</td>
<td>-0.41623005</td>
<td>-0.41614684</td>
<td>-0.0000832134</td>
</tr>
</tbody>
</table>

**Table 3.4:** Results of Euler’s Method

From Table 3.4, we see that dividing \( \Delta t \) by 10 results in an error that is approximately divided by 10, meaning one more digit of accuracy. This confirms that Euler’s method is first order convergent. This observation is more true for the smaller values of \( \Delta t \) than the larger values, illustrating the importance of the qualifier “when \( \Delta t \) is sufficiently small” in the definition of order of convergence.

Notice in Example 3.5.3 that when we went from \( \Delta t = 0.002 \) to \( 0.0002 \), the first three digits in the estimate did not change. This indicates that the estimate \( x(2) \approx -0.41623005 \) is accurate to three digits. Since we know the exact answer, we can confirm that this is indeed true. This observation is important because numerical methods are most useful when exact results are not possible. We can estimate how accurate a numerical result is by reducing the step size a certain amount, and see how many digits of agreement there are between consecutive estimates. We continue to reduce the step size until the desired number of digits of accuracy is reached. The amount by which we reduce the step size depends on the order of convergence.
Higher-Order Numerical Methods

Euler’s method is useful mainly because it is easy to understand why and how it works, and it helps us understand concepts such as step size and how step size relates to order of convergence. However, it is not useful for solving practical problems because it is too slow to converge, meaning that it takes too many steps to yield an accurate estimate. Euler’s method is “training wheels” for learning how to use more sophisticated numerical methods.

In Section 2.4 we introduced two higher-order methods for first-order equations: Modified Euler (also known as Second-order Runge-Kutta) and Fourth-order Runge-Kutta, abbreviated RK2 and RK4. Both of these methods can be easily extended to systems of DE’s in a similar way as with Euler’s method. We will not give the specific formulas for calculating with the Runge-Kutta methods. Instead, we use software to perform the calculations and analyze the results.

**Example 3.5.4** Consider again the mass-spring system in Example 3.5.2. Use the RK4 method on available software to estimate $x(2)$ accurate to six decimal places and justify your claim of accuracy without reference to the exact solution.

**Solution.** We write the IVP in as a system as done in Example 3.5.2 and then use the applet at http://uhaweb.hartford.edu/rdecker/mathlettoolkit/systemsbook.htm to perform the calculations. We select the method used in the Method drop-down box. We start with a step size of $\Delta t = 0.2$. Since RK4 is fourth-order, dividing the step size by 2 should result in an error divided by $2^4 = 16$, resulting in at least one more digit of accuracy. The results are shown in Table 3.5.

<table>
<thead>
<tr>
<th>Step size $\Delta t$</th>
<th>Number of steps $n$</th>
<th>RK4 estimate $x(2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>10</td>
<td>-0.4161210938</td>
</tr>
<tr>
<td>0.1</td>
<td>20</td>
<td>-0.4161452687</td>
</tr>
<tr>
<td>0.05</td>
<td>40</td>
<td>-0.4161467401</td>
</tr>
<tr>
<td>0.025</td>
<td>80</td>
<td>-0.4161468306</td>
</tr>
</tbody>
</table>

**Table 3.5:** Results of RK4 Method

In Table 3.5 we see that when we went from $\Delta t = 0.05$ to 0.025, the first six digits of the estimate of $x(2)$ did not change. This indicates that $x(2) \approx -0.416147$ is accurate to six digits. Note that this estimate is better than what we got using Euler’s method with $n = 10,000$ steps! The higher order of convergence for RK4 makes a huge difference in how quickly we get to a fairly accurate estimate.
In the next example, we look at a variation of the mass-spring system used in the previous examples and illustrate what can go wrong when the step size of a numerical method is not small enough.

**Example 3.5.5** Consider a driven mass-spring system described by the IVP

\[ x'' + x = \sin(0.9t), \quad x(0) = 0, \quad x'(0) = 0. \]

Approximate the response of the system over the interval \(0 \leq t \leq 50\). Use Euler’s method on available software with a step size of \(\Delta t = 0.1\). Describe the response in terms of the mass-spring system. Then use a step size of 0.01 and compare the results.

**Solution.** We use the applet at [http://uhaweb.hartford.edu/rdecker/mathlettoolkit/systemsbook.htm](http://uhaweb.hartford.edu/rdecker/mathlettoolkit/systemsbook.htm). First we convert this IVP into a system by letting \(y = x'\):

\[
\begin{align*}
x' &= y \\
y' &= -x + \sin(0.9t) \\
x(0) &= 1, \quad y(0) = 0.
\end{align*}
\]

The graph of the approximate response using Euler’s method with a step size of \(\Delta t = 0.1\) is shown in Figure 3.21. From the graph, it appears that the amplitude is constantly increasing. This suggests the system is experiencing resonance.

![Graph of the approximate response using Euler's method with \(\Delta t = 0.1\)](image)

**Figure 3.21:** Approximate response using Euler’s method with \(\Delta t = 0.1\)

Repeating the calculations using a step size of \(\Delta t = 0.01\) results in the graph shown in Figure 3.22. Notice that in this graph, the amplitude increases and then decreases. This suggests that the system is experiencing beats, not resonance.
So is the system experiencing resonance or beats? Without finding the explicit solution, we must use a smaller step size. Using a step size of $\Delta t = 0.001$ results in a graph very similar to that in Figure 3.22 (the reader should verify this). This makes us confident in concluding that the system is experiencing beats, not resonance.

In general, this example illustrates that we must be very careful when interpreting the results from numerical methods. We should always confirm our conclusions by using smaller step sizes.

Exercises

**Directions:** Each of the following problems describes a driven mass-spring system. For each problem:

a. Use any available software or the applet at [http://uhaweb.hartford.edu/rdecker/MathletToolkit/SystemsBook.htm](http://uhaweb.hartford.edu/rdecker/MathletToolkit/SystemsBook.htm) to estimate the value of the indicated quantity using the given method and step size.

b. Find the exact value of the indicated quantity and compare the estimate to the exact value.

c. Explain any unusual results you find.

3.5.1 $m = 1, c = 3, k = 2$, driving function $f(t) = 2 \cos(3t)$, initial conditions $x(0) = 1, x'(0) = 0$. Estimate $x(1)$ using Euler’s method with step size 0.25.

3.5.2 $m = 1, c = 2, k = 2$, driving function $f(t) = 3 \cos(t)$, initial conditions $x(0) = 1, x'(0) = 0$. Estimate $x(1)$ using Euler’s method with step size 0.25.
3.5.3 \( m = 1, c = 2, k = 1 \), driving function \( f(t) = -\sin(2t) \), initial conditions \( x(0) = 0, x'(0) = -1 \). Estimate \( x(1) \) using Euler’s method with step sizes \( h = 1.0, 0.1, 0.01, 0.001 \). Based on these calculations, about how many decimals of accuracy do we have with \( h = 0.001 \)?

3.5.4 \( m = 1, c = 0, k = 4 \), driving function \( f(t) = -2\sin(t) \), initial conditions \( x(0) = 0, x'(0) = 0 \). Estimate \( x(1) \) using Euler’s method with step sizes \( h = 1.0, 0.1, 0.01, 0.001 \). Based on these calculations, about how many decimals of accuracy do we have with \( h = 0.001 \)?

3.5.5 \( m = 1, c = 0, k = 4 \), driving function \( f(t) = 2\cos(2t) \), initial conditions \( x(0) = 0, x'(0) = 0 \). Estimate \( x(1) \) using fourth order Runge-Kutta (RK4) with step sizes \( h = 1.0, 0.5, 0.25, \) and \( 0.1 \). Based on these calculations, about how many decimals of accuracy do we have with \( h = 0.1 \)?

3.5.6 \( m = 1, c = 0, k = 100 \), driving function \( f(t) = 40\cos(9t) \), initial conditions \( x(0) = 0, x'(0) = 0 \). Estimate \( x(1) \) using fourth order Runge-Kutta (RK4) with step sizes \( h = 1.0, 0.5, 0.25, \) and \( 0.1 \). Based on these calculations, about how many decimals of accuracy do we have with \( h = 0.1 \)?

**Introduction to Exercises 3.5.7 - 3.5.8:** Some software use numerical methods which use an *adaptive step size*. This means that the algorithm changes the step size as it moves forward in order to stay within a specified maximum error per step. A variation of the fourth-order Runge-Kutta method that uses this idea is called Runge-Kutta-Fehlberg (RKF45). With this type of method, instead of specifying the step size, we specify an error tolerance. On the TI-89 calculator, when the RK method is chosen, the algorithm used is an adaptive step size variation of the RK4 method presented in this section. In this algorithm, we lower the quantity called *diftol* by powers of ten until the desired number of digits of accuracy is obtained. Use this algorithm, or an equivalent one, in Exercises 3.5.7 - 3.5.8.

3.5.7 \( m = 1, c = 0, k = 4 \), driving function \( f(t) = 2\cos(2t) \), initial conditions \( x(0) = 0, x'(0) = 0 \). Estimate \( x(1) \) using an adaptive step size algorithm with error tolerance \( 0.1, 0.01, 0.001 \), and \( 0.0001 \). Based on these calculations, about how many decimals of accuracy do we have with error tolerance \( 0.0001 \)?

3.5.8 \( m = 1, c = 0, k = 100 \), driving function \( f(t) = 40\cos(9t) \), initial conditions \( x(0) = 0, x'(0) = 0 \). Estimate \( x(1) \) using an adaptive step size algorithm with error tolerance \( 0.1, 0.01, 0.001 \), and \( 0.0001 \). Based on these calculations, about how many decimals of accuracy do we have with error tolerance \( 0.0001 \)?
Chapter 3

3.6 Qualitative Methods and the Phase Plane

We now turn to methods for obtaining information from second-order DE’s of the form

\[ x'' = f(x, x') \]

without solving them explicitly (as in Sections 3.1 and 3.3), or numerically (as in Section 3.5). To do so, we will draw graphs called vector fields, direction fields, phase plots, and phase portraits. These are extensions of slope fields and portraits defined for first-order equations in Chapter 2.

The Phase Plane

In Section 3.5 we applied Euler’s method to second-order IVP’s. In each step of the algorithm, we needed to calculate a value of \( t \), a value of \( x \), and also a value of \( x' \) (called \( y \) in the algorithm). This illustrates that we can think of a solution to a second-order IVP as a list of corresponding \( t \), \( x(t) \), and \( x'(t) \) values.

Once we have this list of values, we could plot \( x \) vs \( t \) (\( x \) on the vertical axis, \( t \) on the horizontal axis) or \( x' \) vs \( t \). Such graphs are called time plots. We could also plot \( x' \) vs \( x \). This graph is called a phase plot. The coordinate plane on which a phase plot is drawn is called a phase plane. The next example illustrates these ideas.

Example 3.6.1 Find an explicit solution to each of the following IVP’s. Use the explicit solution to find an explicit description of \( x'(t) \). Then sketch both time plots and the phase plot. Use the interval \( 0 \leq t \leq 20 \). Interpret these graphs in terms of mass-spring systems.

a. \( x'' + 4x = 0 \), \( x(0) = 0 \), \( x'(0) = 2 \)

b. \( x'' + 0.2x' + 4.01x = 0 \), \( x(0) = 0 \), \( x'(0) = 2 \)

c. \( x'' + 4x = \sin(1.6t) \), \( x(0) = 0 \), \( x'(0) = 0 \)

Solution. Each IVP can be solved for \( x(t) \) using the methods of sections 3.1 and 3.3. Then we simply differentiate \( x(t) \) to find \( x'(t) \). The reader can verify the following solutions to each IVP and the calculations of \( x' \):

a. \( x = \sin(2t) \), \( x' = \frac{d}{dt} \sin(2t) = 2 \cos(2t) \)
b. $x = e^{-0.1t} \sin(2t)$, $x' = \frac{d}{dt} \left( e^{-0.1t} \sin(2t) \right) = e^{-0.1t} \left( 2 \cos(2t) - 0.1 \sin(2t) \right)$

c. $x = \frac{25}{36} \sin(1.6t) - \frac{5}{9} \sin(2t)$,  
$x' = \frac{d}{dt} \left( \frac{25}{36} \sin(1.6t) - \frac{5}{9} \sin(2t) \right) = \frac{10}{9} \cos(1.6t) - \frac{10}{9} \cos(2t)$

To generate the time plots, we simply graph $x(t)$ vs $t$ and then $x'(t)$ vs $t$. To generate the phase plot, we could generate a table of values for $t$, $x(t)$, and $x'(t)$, plot the points $(x, x')$, and then draw a smooth curve. This would be extremely tedious and not very practical. A more practical solution is to use the parametric graphing capabilities of available software. In the software, enter the formula for $x$ as $x(t)$. Then enter the formula for $x'$ as $y(t)$.

Figure 3.23: Plots for $x = \sin(2t)$, $x' = 2 \cos(2t)$

The plots for IVP a. are shown in Figure 3.23. If we interpret this system as describing a mass-spring system, these graphs tell us the following:

1. The $x(t)$ time plot shows that the mass starts at its rest position, moves one unit to the right, moves back to rest position, one unit to the left, back to rest position, and the process repeats. This describes an undamped system. From the graph we can also estimate the times at which the object is at rest position and other key positions. For instance, the graph shows that at time $t \approx 4$, the object is 1 unit to the right of rest position.

2. The $x'(t)$ plot tells us about the object’s velocity. The object starts with a velocity of $+2$. The velocity then decreases to 0 and then to $-2$. Then the velocity returns to $+2$ and the process repeats. As with the $x(t)$ time plot, we can approximate the times at which the object’s velocity is at key values. For instance, we see that at time $t \approx 4$, the velocity is 0.

3. The phase plot combines the information from the two time plots. The $x-x'$ plane is called the phase plane, and the curve in the graph is called a trajectory. We can think
of the system as “starting” at the point (0, 2), corresponding to the initial conditions $x(0) = 0$ and $x'(0) = 2$. The system then “moves” down and to the right along the trajectory, indicating that $x$ increases while $x'$ decreases. We can follow the trajectory to see how $x$ and $x'$ change over time. Eventually we get back to where we started and the process repeats. One drawback is that the phase plot does not tell us the times at which the system is at key points. For instance, the graph does not indicate the time at which the system is at the point $(1, 0)$. We need to use the time plots to estimate this time.

The plots for IVP b. are in Figure 3.24. The $x(t)$ time plot shows that the amplitude of the oscillation decreases over time. This shows that the system is under-damped. The $x'(t)$ time plot shows that the velocity oscillates with decreasing amplitude. The phase plot shows that the system “spirals” in a counter-clockwise direction toward the origin, at which point the object would be at rest position and not moving.

![Figure 3.24: Plots for $x = e^{-0.1t} \sin(2t)$, $x' = e^{-0.1t} (2 \cos(2t) - 0.1 \sin(2t))$](image)

The plots for IVP c. are in Figure 3.25. The time plots indicate beats, which show the system is undamped and driven with a driving frequency close to the natural frequency. The phase plot consists of a trajectory that winds around the origin in a clockwise direction, moving in and out while crossing over itself. Note that this system is non-autonomous, due to the $\sin(1.6t)$ term on the right-hand side. This explains the fact that the trajectory crosses itself.

One important point to note is that the phase plot gives us an idea what the $x(t)$ time plot looks like. In other words, if all we had were the phase plot, we would have an idea what the graph of the solution to the IVP looks like. The elliptical shape of the trajectory in Figure 3.23 means that the solution $x(t)$ oscillates. The “inward spiral” shape of the trajectory in Figure 3.24 means that the solution $x(t)$ oscillates with a decreasing amplitude. These behaviors are confirmed by the $x(t)$ time plots.
Another important point is that a trajectory corresponds to the initial conditions. In part a. of Example 3.6.1, the initial conditions were \( x(0) = 0 \) and \( x'(0) = 2 \), and the trajectory went through the point \((0, 2)\). If we were to change the initial conditions to \( x(0) = -1 \) and \( x'(0) = 3 \), for instance, then the trajectory would go through the point \((-1, 3)\).

To better understand the connection between the time and phase plots, notice that we are, in effect, looking at three different views of a parametric curve in 3-dimensional \( t - x - x' \) space.

Figure 3.26: Three views of the space curve \((t, e^{-0.1t} \sin(2t), e^{-0.1t}(2 \cos(2t) - 0.1 \sin(2t)))\)

Figure 3.26 shows a software-generated “space curve” for IVP b. in Example 3.6.1. This curve is generated by plotting the 3-dimensional parametric curve

\[
(t, x, x') = (t, e^{-0.1t} \sin(2t), e^{-0.1t}(2 \cos(2t) - 0.1 \sin(2t)))
\]

with \( t \) as the parameter. If the coordinate axes are rotated so only the \( t \) and \( x \) axes are visible (that is, the \( x' \) axis is perpendicular to the paper), we will see the \( x(t) \) time plot shown in Figure 3.24. Similarly, the other two plots in Figure 3.24 are obtained by rotating the space curve so that the \( x \) axis disappears, resulting in the \( x'(t) \) time plot, or so that
the $t$ axis disappears, resulting in the $x - x'$ phase plot. The middle and right views in Figure 3.26 show how the space curve approaches the $x'(t)$ time plot and the phase plot, respectively, as the axes are turned.

### Parametric Curves and Tangent Vectors

In Example 3.6.1, we drew the plots by first finding an explicit solution. For complicated DE’s, finding an explicit solution may be extremely difficult, if not impossible. So we need a way of approximating the plots without having to find an explicit solution. To do this, we first review the idea of a parametric curve.

A parametric curve is a curve on the $x$-$y$ plane in which the $x$- and $y$-coordinates are described in terms of a variable called a parameter, typically denoted with the letter $t$ (note that this is a different use of the term than we have previously used). In generic terms, a parametric curve is described with equations of the form

$$x = f(t) \quad \text{and} \quad y = g(t).$$

Now consider a point $(f(t_0), g(t_0))$ on a parametric curve. Recall, from the calculus of several variables, that the vector

$$f'(t_0) \mathbf{i} + g'(t_0) \mathbf{j},$$

where $\mathbf{i}$ is the unit vector along the $x$-axis and $\mathbf{j}$ is the unit vector along the $y$-axis, is tangent to the curve at this point. This vector is called a tangent vector and describes the direction in which the curve is being traced out at this point. If the curve represents the position of an object, then the tangent vector is called a velocity vector. The direction of the velocity vector indicates the direction the object is moving at time $t_0$, and the magnitude of the velocity vector is the speed of the object.

**Example 3.6.2** Find the tangent vector to the parametric curve

$$x = e^{-t/10} \cos(t), \quad y = e^{-t/10} \sin(t).$$

at the point where $t = \pi/3$.

**Solution.** The graph of this curve over the interval $0 \leq t \leq 6\pi$ is shown in Figure 3.27. Note the inward spiral shape.

The $x$- and $y$-coordinates of the point corresponding to $t = \pi/3$ are

$$x = e^{-\pi/30} \cos(\pi/3) \approx 0.45, \quad \text{and} \quad y = e^{-\pi/30} \sin(\pi/3) \approx 0.78.$$
To find the tangent vector at this point, we first calculate the derivatives of $x$ and $y$ with respect to $t$:

$$x' = \frac{d}{dt} e^{-t/10} \cos(t) = e^{-t/10} \left(-\frac{1}{10} \cos(t) - \sin(t)\right),$$

$$y' = \frac{d}{dt} e^{-t/10} \sin(t) = e^{-t/10} \left(\cos(t) - \frac{1}{10} \sin(t)\right).$$

Next we evaluate these derivatives at $t = \pi/3$ and obtain $x' \approx -0.825$ and $y' \approx 0.372$. Thus the tangent vector is approximately

$$-0.825 \hat{i} + 0.372 \hat{j}.$$  

To plot this tangent vector, we put the tail at the point $(0.45, 0.78)$. The head of the vector is plotted $-0.825$ units to the left and $0.372$ units up from the tail. Figure 3.27 shows this tangent vector along with a number of other tangent vectors.

**Autonomous Systems and Second-order Equations**

As for the case of first-order differential equations, we call a system of first-order equations, or a second-order equation, *autonomous* if the independent variable (which we will call $t$) does not appear explicitly. For example, the second-order DE $x'' + 2x' + 2x = 0$ and its corresponding first-order system $x' = y, y' = -2x' - 2x$ are autonomous, whereas the second-order DE $x'' + 2x' + 2x = \sin(t)$ and its corresponding first-order system $x' = y,$
$y' = -2x' - 2x + \sin(t)$ are nonautonomous.

For the rest of this section, we will be concerned only with the autonomous case.

**Vector Fields, Direction Fields, and Phase Portraits for Autonomous Equations and Systems**

**Second-order Equations**

From Figure 3.27, we see that if all we had were several tangent vectors, we would have a good idea of the shape of a parametric curve. From Example 3.6.1, we see that if we knew the phase plot of an initial value problem (second-order DE, or first-order system of two DE’s) we could give a rough description of the solution, that is, we could roughly describe the graphs of $x(t)$ vs $t$ and $x'(t)$ vs $t$. Next we show how to get a description of many phase plots at once without actually calculating an explicit solution first.

We know that a phase plot of a second-order IVP is a plot of $x'$ vs $x$. A trajectory is the parametric curve given by

$$x = x(t) \quad \text{and} \quad y = x'(t)$$

where $x(t)$ denotes a solution of the IVP and $x'(t)$ denotes the derivative of $x(t)$. A tangent vector at a generic point on a phase plot is described by

$$x'(t)\vec{i} + x''(t)\vec{j}.$$ 

If we can find and plot several tangent vectors, then we could roughly describe the trajectory. Now consider an autonomous second-order DE of the form $x'' = f(x, x')$. Using this notation, a tangent vector is described by

$$x'\vec{i} + f(x, x')\vec{j}. \quad (3.13)$$

In order to plot tangent vectors, we don’t need to solve the DE, all we need to do is choose a few points on the phase plane, $(x, x')$ and calculate the tangent vectors using equation (3.13). This leads to the following definition.

**Definition 3.6.1** A vector field for an autonomous second-order differential equation $x'' = f(x, x')$ is an array of vectors in the phase plane where the vector at the
point \((x, x')\) is given by
\[
x' i + f(x, x') j.
\]

**First-order Systems**

In a manner similar to that of second-order autonomous DE’s, given an autonomous first-order system
\[
x' = f(x, y), \quad y' = g(x, y)
\]
and solution \((x(t), y(t))\), a tangent vector would be given by \(x'(t)i + y'(t)j\). Thus given a point \((x, y)\), the tangent vector of any solution curve through that point would be given by \(f(x, y)i + g(x, y)j\). This leads to the following definition.

**Definition 3.6.2** A vector field for an autonomous first-order system of differential equations \(x' = f(x, y), \quad y' = g(x, y)\) is an array of vectors in the phase plane where the vector at the point \((x, y)\) is given by
\[
f(x, y)i + g(x, y)j.
\]

Since a second-order differential equation can always be written as a first-order system, given a second-order equation, we can use the approach in either Definition 3.6.1 or in Definition 3.6.2. One advantage of Definition 3.6.2 is that many software programs require that the equation be written in system form.

Tangent vectors can be very long, but one is often only interested in the direction of the vector, not its length. Thus it is usually desirable to make all of the vectors in the field about the same length (but with the same direction as the vector field). This leads to the next definition.
**Definition 3.6.3** A *direction field* is a vector field where each vector is scaled (either shortened or lengthened) in such a way to make the graph easy to read.

A vector field, or direction field, gives us an idea of what one, or many trajectories looks like.

**Example 3.6.3** Consider the autonomous second-order equation \( x'' + 2x' + 2x = 0 \). Sketch both a vector field and a direction field in the region \(-1 \leq x \leq 1, -1 \leq x' \leq 1\) of the phase plane. Use a 3 by 3 array of vectors where the tails of the vectors have coordinates with integer values.

**Solution.** Rewriting the DE in the form \( x'' = f(x, x') \) yields \( x'' = -2x' - 2x \). Thus using Definition 3.6.1 the vectors are given by

\[
x' \vec{i} + (-2x' - 2x) \vec{j}.
\] (3.14)

We could also write the second-order equation as a first-order system using \( y = x' \). The system is \( x' = y, y' = -2x - 2y \), so using Definition 3.6.2 the vectors would be given by the formula

\[
y \vec{i} + (-2x - 2y) \vec{j}.
\] (3.15)

which is the same as Equation (3.14) given that \( y = x' \).

We calculate these vectors at the points \((x, x')\) described by the numbers along the left and bottom sides of Table 3.6. For example when \( x = 0 \) and \( x' = -1 \) we get

\[-1 \vec{i} + [-2(0) - 2(-1)] \vec{j} = -\vec{i} + 2 \vec{j}.
\]

The vectors at the other points are shown in the center of the table.

The resulting vector field is shown in the left-half of Figure 3.28. The associated direction field shown in the right-half of the figure is drawn by simply shortening each arrow to make the graph easier to read (note also that the scales on the axes have changed).

The vector and direction fields indicate that a trajectory for this DE spirals around the origin in a clockwise direction. Whether the trajectory is spiraling toward the origin, or away from the origin, it hard to tell due to the relatively small number of vectors. This spiraling behavior tells us that a solution to the DE, \( x(t) \), will oscillate over time. \( \square \)
Example 3.6.3 illustrates that drawing a direction field by hand is extremely tedious, and that getting useful information from the field requires more than 9 vectors. For this reason, software is typically used to draw direction fields. Along with the direction field, the software usually draws several trajectories corresponding to different pairs of initial conditions. A direction field along with several trajectories is called a phase portrait.

A software-generated phase portrait for the DE $x'' + 2x' + 2x = 0$ (the same equation as in Example 3.6.3) is shown in Figure 3.29. This phase portrait shows that trajectories spiral in a clockwise direction toward the origin. This means that a solution to the DE oscillates with a decreasing amplitude.

Usually software allow the user the option to plot only one trajectory, corresponding to one pair of initial conditions, and then draw the corresponding time plots. The trajectory and time plots for the initial conditions $x(0) = 9$ and $x'(0) = 4$, for example, are shown in Figure 3.30. Note that $x(t)$ does indeed oscillate with decreasing amplitude.

Several comments about phase portraits in general are warranted:

1. The trajectories are calculated by first selecting a pair of initial conditions, then a numerical method such as Euler’s method or RK4 is used to approximate a numerical solution. The resulting points $(x, x')$ are plotted on the phase plane, and the points
Figure 3.29: Phase portrait for $x'' + 2x' + 2x = 0$

Figure 3.30: Phase plane and time plots for $x(0) = 9$ and $x'(0) = 4$

are connected with a smooth curve. The resulting trajectory is not exact, as those in Example 3.6.1, but rather they are approximations.

2. The number of trajectories drawn (or equivalently the number of different pairs of initial conditions selected) depends on the DE. We want enough trajectories to give a good picture of the solution over a wide variety of initial conditions, but not too many as to clutter up the portrait.

3. Some software require that the second-order DE first be converted to a system of two first-order equations. This can very easily be done when the DE is of the form $x'' = f(x, x')$. If we define $y(t) = x'(t)$, then the equivalent system is

$$x' = y, \quad y' = f(x, y).$$
4. Some software plot the vector corresponding to the point \((x, x')\) so that the point is at the midpoint of the vector rather than the tail of the vector.

5. Phase portraits only make sense for autonomous DE's. If the equation were non-autonomous and of the form \(x'' = f(x, x', t)\), where the right-hand side depends explicitly on \(t\), then a tangent vector would be described by

\[
x'i + f(x, x', t)j.
\]

The fact that \(t\) explicitly appears in this description means that a vector would change over time. In practical terms, this means that a vector on the phase plane would not stay still, it would rotate as we look at it. Also, the trajectories would move as we look at them. It would be very difficult to gain any useful information from such a graph.

### Existence and Uniqueness

Figure 3.29 is a phase portrait for an autonomous second-order DE. Observe that trajectories never cross, and it appears that through any point on the phase plane, we could draw a trajectory. Since each point on the phase plane corresponds to a pair of initial conditions, these observations indicate that a second-order IVP has a unique solution. This conclusion is formalized in the following theorem, which is a generalization of Theorem 3.1.

**Theorem 3.6 (Existence/Uniqueness for Autonomous Second-Order IVP’s)** Consider an IVP of the form

\[
x'' = f(x, x'), \quad x(t_0) = x_0, \quad x'(t_0) = x'_0.
\]

If the functions

\[
f(x, x'), \quad \frac{\partial f}{\partial x}, \quad \text{and} \quad \frac{\partial f}{\partial x'},
\]

are all continuous in some rectangle \(R = \{a < x < b, c < x' < d\}\) in the phase plane, and if the point \((x_0, x'_0)\) is in \(R\), then there exists a unique solution to the IVP which is continuous for all \(t\) in some (possibly very small) interval around \(t = t_0\).

If the function \(f\) and its derivatives are continuous everywhere in the phase plane, then each trajectory in a phase portrait becomes a kind of fence. Theorem 3.6 says that trajectories, those drawn and those that we could draw, cannot intersect. By filling out the phase plane with a sufficient number of trajectories (with carefully chosen initial conditions) one gets a “map” of the differential equation, which shows how the solution through any pair of initial conditions will move and where it will “end up.”
CHAPTER 3  Second-order Differential Equations

It is important to understand that these ideas do not apply to time plots, even if the equation is autonomous. Figure 3.31 shows the phase portrait from Example 3.6.3 containing trajectories corresponding to six different pairs of initial conditions. Also shown are the time plots corresponding to the same pairs of initial conditions. We see that the trajectories in the phase portrait do not intersect, but that the curves in the time plots do intersect.

Figure 3.31: Phase portrait and time plots for \( x'' + 2x' + 2x = 0 \)

Nonlinear Equations and Fixed Points

The real power of phase portraits comes into play when analyzing nonlinear DE’s. Because nonlinear equations are, in general, much harder to solve than linear ones, qualitative methods such as the phase portrait, and numerical methods such as Runge-Kutta are often the only means of obtaining information about the solution(s).

Second-order Equations

To illustrate how we can analyze a nonlinear second-order equation with a phase portrait, consider the DE

\[
\frac{d^2\theta}{dt^2} + c\frac{d\theta}{dt} + k\sin(\theta) = 0
\]

which models the motion of a damped, rigid pendulum, where \( \theta \) measures the angle (in radians, measured in a counter-clockwise direction) that the pendulum makes with a vertical line as illustrated in Figure 3.32. The parameter \( c \) measures the damping due to friction and \( k \) takes into account the mass and length of the pendulum.

A phase portrait for this DE with \( c = 0.5 \) and \( k = 1 \) is shown in Figure 3.33.

Consider the points \((0, 0)\) and \((\pi, 0)\) on the phase plane in Figure 3.33. Notice that the trajectories spiral toward the point \((0, 0)\) and are “repelled,” or move away, from the point \((\pi, 0)\). These observations beg two questions:
3.6 Qualitative Methods and the Phase Plane

1. What is significant about these points in terms of the DE?
2. What is significant about these points in terms of the motion of the pendulum?

At the point \((0, 0)\), we have \(\theta = 0\) and \(\theta' = 0\). The fact that \(\theta = 0\) means that the pendulum is at the bottom of its swing. The fact that \(\theta' = 0\) means that the pendulum has no angular velocity. Next note that we can rewrite the DE into the form

\[
\frac{d^2 \theta}{dt^2} = -0.5 \frac{d\theta}{dt} - \sin(\theta). \tag{3.16}
\]

At the point \((0, 0)\), we have

\[
\frac{d^2 \theta}{dt^2} = -0.5(0) - \sin(0) = 0 - 0 = 0.
\]

This means that the pendulum has no angular acceleration at this point. With no angular velocity or acceleration, the pendulum has stopped moving and will not start moving again. The fact that trajectories spiral toward the point \((0, 0)\) once they get “close” to the point...
(0, 0) physically means that once the pendulum gets close to the bottom of its swing, and with a small enough velocity, it will remain close to the bottom of its swing. Hopefully this agrees with our intuition.

Now consider the point (π, 0). The fact that θ = π means that the pendulum is at the top of its swing. Also we have θ′ = 0, again meaning that the pendulum has no angular velocity. Using equation (3.16), we also have

\[
\frac{d^2 \theta}{dt^2} = -0.5(0) - \sin(\pi) = 0 - 0 = 0,
\]

again meaning that the pendulum has no angular acceleration and thus has stopped moving. The fact that trajectories are repelled from the point (π, 0) once they get close to the point, physically means that if the pendulum gets close to the top of its swing, and with any angular velocity, then it will move away from the top of its swing. Again, hopefully this agrees with our intuition.

These ideas are illustrated in Figure 3.34 which shows the trajectory for a pendulum which almost comes to rest at the top of its swing, corresponding to the point (π, 0), but has just enough energy to get over the top, and ends up oscillating about, and ultimately approaching, the rest position corresponding to the point (2π, 0).

![Pendulum phase plot](image)

**Figure 3.34:** Pendulum going over the top

Note that we could do similar analysis with the points (2π, 0), (−π, 0), and (−2π, 0).

Similar arguments apply to examples involving autonomous systems of first-order equations \(x' = f(x, y), \ y' = g(x, y)\). In this case a fixed point would occur when both \(f(x, y) = 0\) and \(g(x, y) = 0\). These observations lead to the following definitions.
Definition 3.6.4 For an autonomous second-order DE of the form \( x'' = f(x, x') \), a fixed point (also called an equilibrium point or critical point) is a point \((\bar{x}, 0)\) on the phase plane at which
\[
f(\bar{x}, 0) = 0.
\]

Definition 3.6.5 For a system of two first-order DE’s of the form \( x' = f(x, y) \), \( y' = g(x, y) \) a fixed point (also called an equilibrium point or critical point) is a point \((\bar{x}, \bar{y})\) on the phase plane at which
\[
f(\bar{x}, \bar{y}) = 0 \text{ and } g(\bar{x}, \bar{y}) = 0
\]

Definition 3.6.6 A fixed point (for either a second-order equation or a system of two first-order equations) is said to be stable if trajectories move toward the point once they get “close enough.” A fixed point is said to be unstable if trajectories move away from the point once they get close enough.

Example 3.6.4 Find all the fixed points of the DE \( \theta'' = -0.5\theta' - \sin \theta \) and graphically classify each as stable or unstable.

Solution. Note that we have \( f(\theta, \theta') = -0.5\theta' - \sin(\theta) \). To find all the fixed points of this equation, we can use Definition 3.6.4 and solve the algebraic equation
\[
0 = f(\bar{\theta}, 0) = -0.5(0) - \sin(\bar{\theta}) = -\sin(\bar{\theta}).
\] (3.17)
We could also write the DE as a first-order system and use Definition 3.6.5. Letting \( x = \theta \)
and \( y = \theta' \) we get the DE’s \( x' = y \) and \( y' = -0.5y - \sin(x) \). Solving the system \( \bar{y} = 0, -0.5\bar{y} - \sin(\bar{x}) \) leads to \( \sin(\bar{x}) = 0 \) which is equivalent to Equation (3.17) given that \( x = \theta \). The solutions to this equation are \( \theta = n\pi \), where \( n \) is an integer (positive, negative, or zero). From Figure 3.33, it appears that the fixed points corresponding to even values of \( n \) are stable, and those corresponding to odd values are unstable.

Fixed points will be studied more formally in the context of systems of DE’s in Chapter 5.

**Exercises**

3.6.1 In part a. of Example 3.6.1, show that the trajectory in the phase plot is an ellipse described by the equation

\[
 x^2 + \left( \frac{x'}{2} \right)^2 = 1.
\]

For problems 1-4, solve each initial value problem, and display the exact solution in the phase plane and also as \( x(t) \) and \( x'(t) \) time plots. Use \( 0 \leq t \leq 5 \), and choose appropriate intervals for \( x \) and \( x' \). Give an interpretation of each equation and its solution as a mass-spring system.

1. \( x'' + 7x' + 12x = 0, \quad x(0) = 0, \quad x'(0) = 10. \)
2. \( x'' + 7x' + 12x = 3\cos 3t, \quad x(0) = 0, \quad x'(0) = 10. \)
3. \( x'' + 2x' + 17x = 0, \quad x(0) = 5, \quad x'(0) = 0. \)
4. \( x'' + 2x' + 17x = 20\sin 10t, \quad x(0) = 5, \quad x'(0) = 0. \)

For problems 5-8, redo problems 1-4, but this time generate the plots (phase plot and two time plots) using a numerical method as in Figure 3.34 (do not create an entire phase portrait, just the plots for the one solution curve corresponding to the IVP). State which numerical method you are using, and any other relevant settings such as step size or error tolerance. Use whatever technology you have available, or you can use the applet for graphing and calculating numerical solutions for systems of equations at [uhaweb.hartford.edu/rdecker/MathletToolkit/SystemsBook.htm](http://uhaweb.hartford.edu/rdecker/MathletToolkit/SystemsBook.htm).

5. Use the IVP from problem 1.
6. Use the IVP from problem 2.
7. Use the IVP from problem 3.

8. Use the IVP from problem 4.

For problems 9-12, sketch a direction field for each autonomous second-order differential equation by hand, using a 3 by 3 array of vectors in the region $-1 \leq x \leq 1$, $-1 \leq x' \leq 1$ as in Example 3.6.3.

9. Use the DE from problem 1.

10. Use the DE from problem 3.

11. $x'' - 5x' + 6x = 0$.

12. $x'' + x' - 6x = 0$.

For problems 13-16 create a phase portrait for the given linear autonomous equation in the region $-5 \leq x \leq 5$, $-20 \leq x' \leq 20$. It should include a direction field and enough well-chosen solution curves to give a complete picture of the equation on the specified region. From the picture, determine whether the fixed point at the origin is stable or unstable. Use whatever technology you have available, or you can use the applet for graphing and calculating numerical solutions for systems of equations at uhaweb.hartford.edu/rdecker/MathletToolkit/SystemsBook.htm.

13. Use the DE from problem 1.

14. Use the DE from problem 3.

15. $x'' - 5x' + 6x = 0$.

16. $x'' + x' - 6x = 0$.

For problems 17-18 create a phase portrait for the given nonlinear autonomous equation in the given region. Also find any fixed points that occur in that region. Your phase portrait should include a direction field and enough well-chosen solution curves to give a complete picture of the equation on the specified region, especially near the fixed points. From the picture, determine which fixed points are stable and which are unstable. Use whatever technology you have available, or you can use the applet for graphing and calculating numerical solutions for systems of equations at uhaweb.hartford.edu/rdecker/MathletToolkit/SystemsBook.htm.

17. $y'' + 2y' - 17(4y - y^3) = 0$ on $-5 \leq y \leq 5$, $-20 \leq y' \leq 20$ (this equation can be used as a model of a mass-spring system with a repelling force at the origin - explain why, using your phase portrait).
18. \( \frac{d^2\theta}{dt^2} + 2 \frac{d\theta}{dt} + \sin \theta = 0 \) on \(-7 \leq \theta \leq 7, -1 \leq \theta' \leq 1\) (note that this is the pendulum equation, with different parameters, from Example 3.6.4 - relate the behavior of the pendulum to your phase portrait).

### Lab 2: Nonlinear forces and the Duffing equation

#### Introduction

We have spent a great deal of time studying the linear mass-spring equation \( m\ddot{x} + c\dot{x} + kx = f(t) \), where \( x \) represents the displacement from equilibrium of an object of mass \( m \) suspended from a linear spring with spring constant \( k \) and damping constant \( c \). The independent variable is \( t \) which represents the time. The forcing function \( f(t) \) (when it is not zero) has often been assumed to be of the form \( a \cos(\omega t) \) or \( a \sin(\omega t) \). The term \( kx \) represents the force, as a function of \( x \), for a linear spring; if we replace that term with a nonlinear function of \( x \) we get a nonlinear force on the mass. For some nonlinear forces this can still be considered to be a mass-spring system (see the investigation of hard and soft springs below), but for other forces a different physical model is needed.

One of the most-studied, and simplest, examples of a nonlinear mass-spring equation is the Duffing equation

\[
mx'' + cx' + kx + bx^3 = f(t).
\]

Here the force on the mass is given by the cubic function \( kx + bx^3 \). When written as a system of first-order differential equations, using the usual substitution \( y = x' \), we get \( x' = y \) and \( y' = -cy - kx - bx^3 + f(t) \). Two good websites that give an introduction to this very interesting equation are http://www.scholarpedia.org/article/Duffing_oscillator and http://mathworld.wolfram.com/DuffingDifferentialEquation.html (the letters used for the various parameters are not completely standard, so be careful when applying the information from those websites to this project).

Clearly the only difference between the Duffing equation and the standard mass-spring equation is the replacement of the linear term \( kx \) by the nonlinear term \( kx + bx^3 \). This term therefore describes the force on the mass when it is displaced \( x \) units from its equilibrium position. Surprisingly, this simple looking term can describe a number of different physical systems, depending on the choice of parameters \( k \) and \( b \). Note on the direction of the force: For a linear mass-spring system, the derivation of the equation \( mx'' + cx' + kx = f(t) \) comes from Newton’s second law, which says that the mass of an object multiplied by its acceleration is equal to the sum of all forces acting on the object. For the mass-spring
system this becomes $m x'' = -c x' - k x + f(t)$. The negative sign on the $k x$ term is there so that when the spring is extended in the direction of positive $x$, the force acts in the opposite direction in order to restore the spring to its rest position (a similar argument works for the damping force $c x'$). Thus keep in mind for activities below, that when you graph the force $k x + b x^3$, one expects that for a spring the graph would be positive (above the $x$-axis) when $x$ is positive, and negative (below the $x$-axis) when $x$ is negative.

Preliminaries: Hard and soft springs

First we want to understand what is meant by the terms “soft spring” and “hard spring”.

1. Graph the linear and nonlinear spring-force functions $k x$ and $k x + b x^3$ together for $k = 1$ and $b$ varying from $-3$ to $5$. First compare the two graphs near the origin (for this choose a graph window that is something like $-0.1$ to $0.1$ in both the horizontal and vertical directions). Is there much difference between the two graphs? What does this imply about the nonlinear model for small oscillations (small displacements $x$ from the origin)? When the displacement $x$ is sufficiently small, we say that the nonlinear spring is in the linear range.

2. Next compare the two graphs further from the origin (say with a graph window $-1$ to $1$ in both directions). Again, let $b$ vary between $-3$ and $5$. In particular notice whether the graph of $k x + b x^3$ is above (stronger force) or below (weaker force) the graph of $k x$. For which values of $b$ is the force of the spring in the nonlinear model greater than that of the linear model for positive $x$? For which values of $b$ is the force less than for the linear model for positive $x$? Explain why the nonlinear spring is called a soft spring when $b < 0$ and a hard spring when $b > 0$. Are these terms appropriate for small oscillations also (remember what you observed in part 1)?

3. Investigate the question “For what range of $x$ values is the force equation $k x + b x^3$ physically realistic?” Consider both hard and soft springs, using the $b$ values and window setting from part 2. In particular, for any sensible spring the force should pull the spring back towards the equilibrium position for any amount of displacement $x$; if the graph of the force goes negative, then the spring is pushing the mass away from the equilibrium position (a repelling force), which a spring would never do. A more subtle consideration would be that as $x$ gets larger, the force should get larger. Otherwise, the spring would feel “tight” for small $x$ and “loose” for large $x$, sort of like a piece of taffy. The problem is that once taffy is stretched out, it does not snap back to its original state as a spring does. Find the values of $x$ for which the spring becomes somewhat physically unrealistic (the “taffy” spring) and which are totally
impossible for a real spring (the repelling spring) in terms of the constants $k$ and $b$. Note: this does not mean that soft springs in general are not possible for large $x$, only that the Duffing equation must be restricted to a certain range of $x$ values in order for it to be used as a model of a soft spring.

**Hard and soft springs and the Duffing equation**

Now that we understand how the term $kx + bx^3$ works as a model for soft and hard springs, we investigate the behavior of the entire mass-spring system (as modelled by the Duffing equation) for soft and hard springs. We will restrict our discussion to the case when the displacements are large enough that the nonlinear spring is no longer in the linear range (see 1 above), and to the case when there is no damping ($c = 0$), and no forcing function ($f(t) = 0$).

1. Obtain graphs of the solution curves to $mx'' + cx' + kx + bx^3 = 0$ for $m = 1$, $c = 0$, $k = 1$, with the initial conditions $x(0) = 0.5$ and $x'(0) = 0$, and let $b$ increase slowly from $-3$ to $5$. Look at both a phase plot (plotting the spring-force function $kx + bx^3$ on top of the phase plot helps relate this exercise with the previous ones) and a time plot. Good ranges for the phase plot are $-1$ to $1$ in both directions, and for the time plot $0$ to $20$ in the horizontal ($t$) direction and $-1$ to $1$ in the vertical ($x$) direction. Time plots reveal the period/frequency of the oscillations. How does the softness or hardness of the spring (the value of $b$) affect the period and frequency? Note anything else of interest that you observe in either the phase or time plots.

2. Using the same graph windows as in part 3, fix $b = -2$ (soft spring) and let the initial position $x(0)$ vary (keeping $x'(0) = 0$). How does the period/frequency depend on $x(0)$? For what initial positions $x(0)$ is the response of the spring physically unrealistic (see part 3 from above)? Repeat for $b = 5$ (hard spring) and answer the same questions. How does the period/frequency depend on $x(0)$ for a linear spring (think about what it means for a linear spring to have a “natural” frequency)? Do nonlinear springs have a natural frequency? Explain.

3. As noted in the introduction, the corresponding first-order system for the Duffing equation is $x' = y$, $y' = -cy - kx - bx^3$ (since we are currently assuming that $f(t) = 0$). Find the fixed points for the system. Now, for $b = -2$ (soft spring) create a phase portrait (many solution curves) and/or a direction field in order to observe the global behavior of the system. Again it is helpful to graph the spring force $kx + bx^3$ on the same set of axes as the phase portrait (the intersections of that curve with the horizontal axis are the fixed points - show why). Carefully describe what the various
possible solution curves represent in terms of the spring. In your discussion include the solution curves that correspond to physically non-realistic springs as discussed in part 3 of “Preliminaries: Hard and soft springs”. Are the fixed points stable or unstable based on the phase portrait? Repeat for $b = 5$ (hard spring) and answer the same questions.

### Beams and magnets and the Duffing equation

Hard and soft springs correspond to what is called a single well potential: this means that there is a single stable fixed point, as we saw in part 3 of “Hard and soft springs and the Duffing equation”. Here we will see that the Duffing equation $m x'' + c x' + k x + b x^3 = f(t)$ can be used to model systems that have two stable fixed points, with an unstable fixed point between them (a double well potential). There are a number of ways of creating such a system. See the first website given in the introduction for an example, where a thin flexible metal beam is suspended between two magnets (the beam is stable when it is bent towards one magnet or the other, but unstable when it is in the middle).

The parameters $m$ and $k$ will no longer have the exact same interpretation as for a mass-spring system, although $m$ will be related to (but not equal to) the mass of the beam. The $c$ parameter can still be interpreted as a damping constant.

1. We first investigate the beam-magnet force given by the term $k x + b x^3$. When applied to the beam-magnet system, $x$ represents the displacement of the tip of the beam. Of course, this is the same function as for the hard and soft springs, but this time we assume that $k$ is negative. Graph $k x + b x^3$ in a graph window with both horizontal and vertical axes running from $-1$ to $1$. Set $k = -1$. Let $b$ vary between $-3$ and $5$. For what values of $b$ does this force function represent a double well potential (a beam-magnet system)? For $b$ values which do not correspond to a double well potential, can you think of a real system which would act like this?

2. Again for $k = -1$, choose a typical $b$ value that represents a double well based on your investigation in part 1. With this $b$ value create a phase plot and a time plot for the unforced, undamped Duffing equation $x'' + k x + b x^3 = 0$ (with $m = 1$). Start with $x(0) = 0.5$ and $x'(0) = 0$, then vary $x(0)$ to see what type of solution curves are possible for these parameter values. Do all solution curves correspond to oscillations of some type? Explain how the solution curves correspond to the motion of the tip of the beam in the beam-magnet system ($x$ is as in part 1). Discuss both the phase plot and the time plot.
3. For \( k = -1 \) and the same \( b \) value and equation as in part 2, create a phase portrait (several solution curves). Adding a direction field can help also. Do not include time plots, as they tend to get “tangled” when many are plotted at once. Now slowly increase \( k \) and continue until it reaches \( k = 1 \). At this point you should be back to the hard spring of the “Hard and soft springs” investigation. Explain what happened to the various fixed points and their stability as \( k \) changed. Also relate what happens to the beam-magnet system. Can you relate the value of \( k \) to any property of the beam-magnet system? Explain.

**Beats, resonance and chaos in the Duffing equation**

Recall that for the undamped linear mass-spring equation \( mx'' + kx = f(t) \) one observes beats when the forcing function is periodic (we will assume the form \( a \sin(\omega t) \)) and the natural frequency (given by \( \sqrt{k/m} \)) is close to the driving frequency (given by \( \frac{\omega}{2\pi} \)). Resonance occurs when the two frequencies are exactly equal. Do we get similar phenomena with nonlinear systems?

1. Demonstrate with time plots how beats turn into resonance and back to beats in the linear, undamped mass-spring equation \( mx'' + kx = a \sin(\omega t) \). Use \( m = 1, k = 1, a = 1, x(0) = x'(0) = 0 \) and let \( \omega \) vary slowly between 0.5 and 1.5. Resonance occurs at \( \omega = 1 \) (explain why). Describe both in words and pictures what happens. Suppose you are observing a mechanical system, such as the vibrations of an airplane wing, and you detect beats. As time passes, the beats are getting farther and farther apart in time, so you think things are getting better. Are they?

2. One can observe beats in nonlinear systems as well. Recall that from the “Beams and magnets” activity above, when \( k \) is negative the Duffing equation can be thought of as a model for a beam-magnet system. For the Duffing equation \( mx'' + kx + bx^3 = a \sin(\omega t) \) with \( k = -1, b = 1, a = 0.07, x(0) = 1, x'(0) = 0 \), let \( \omega \) vary slowly between 1 and 2. Again, describe what happens in terms of beats. Does resonance occur this time? What does occur? Describe as best you can, both in mathematical terms (include a discussion of fixed points) and in terms of the beam-magnet system. Relate this to the information you found at the websites listed in the introduction.
Chapter 4

Linear Algebra Interlude

Linear algebra is a branch of mathematics that can be used to solve problems that involve a large number of variables and a large number of equations. The important requirement for linear algebra to be helpful is that the variables must appear in a linear way in the equations. When that is the case, linear algebra techniques can be used to solve both algebraic equations and differential equations.

Suppose \( x, y, z \) are variables that represent the \( x, y, z \) displacements of one end of a beam in a large structure. In static equilibrium, these variable may satisfy a set of equations such as

\[
\begin{align*}
2x + 3y + 5z &= 10 \\
x + y + 7z &= 12 \\
3x + 7y + 8z &= 5
\end{align*}
\]

Notice that the variables all appear in a linear way (multiplied by a constant, but not raised to a power, no exponential or trig functions of variables, and so on). These equations can then be solved using linear algebra for \( x, y, \) and \( z \). In a real structure, there may be thousands of variables and equations.

We can also describe and solve systems of linear DE’s. For the structure described on the previous slide, if the end of the beam is not in static equilibrium, then we may need differential equations to describe its motion. We might get a system such as

\[
\begin{align*}
x' &= 2x + 3y + 5z \\
y' &= x + y + 7z \\
z' &= 3x + 7y + 8z
\end{align*}
\]
Again, for linear algebra to apply, the system must be linear in all dependent variables. As before, there may be many more than three equations and three dependent variables in a real system (structure, circuit, ecosystem).

### 4.1 Matrices, Vectors, Scalars

A *matrix* is a rectangular array of numbers, usually enclosed by brackets. Generally we use uppercase letters to represent matrices. Here is a matrix with 2 rows and 3 columns:

\[
A = \begin{bmatrix}
0 & 1 & -2 \\
3 & -4 & 5
\end{bmatrix}
\]

We list the number of rows first then the number of columns when describing a matrix. Thus the matrix \(A\) above is referred to as a 2 by 3, or \(2 \times 3\) matrix. Shortly we will see how matrices are used to describe large systems of linear algebraic or differential equations.

A *vector* is just a matrix with one column or one row. We will normally work with vectors that consist of just one column, called column vectors. A row vector would have just one row. We use lower case letters for vectors such as

\[
b = \begin{bmatrix}
1 \\
-2
\end{bmatrix}
\]

A *scalar* is a 1 by 1 matrix, that is, just a single number. Use lower case for scalars, such as \(c = 5\).

**Matrix algebra**

Much of the power of using matrices to describe linear systems, is that you can manipulate them using algebraic operations that mimic the algebraic operations of numbers and variables that you learned in high school.

**Matrix addition**

Matrices of the same size can be added, term by term. Thus a \(2 \times 2\) matrix can be added to another \(2 \times 2\) matrix; just add corresponding entries. If the two matrices are not the
same size, matrix addition is not defined. For example if

\[
A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -1 & 1 \\ 3 & 2 \end{bmatrix}
\]

(4.1)

then

\[
A + B = \begin{bmatrix} 1 - 1 & 2 + 1 \\ 3 + 3 & 4 + 2 \end{bmatrix} = \begin{bmatrix} 0 & 3 \\ 6 & 6 \end{bmatrix}
\]

**Scalar multiplication**

A matrix can be multiplied by a scalar; just multiply each element of the matrix by the scalar. If we define the scalar \( c = 2 \), with \( A \) defined above, then

\[
cA = \begin{bmatrix} 2(1) & 2(2) \\ 2(3) & 2(4) \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}
\]

**Matrix multiplication**

In certain cases, matrices can be multiplied. First we define the product of a column vector and a row vector, where both are the same length. If \( c \) is a row vector and \( d \) is a column vector of the same length, the product \( cd \) is a scalar that results from multiplying corresponding entries (first entry of \( c \) times first entry of \( d \), then second entry of \( c \) times second entry of \( d \), etc.), then adding the results. Thus for the vectors

\[
c = \begin{bmatrix} 1 & 3 \end{bmatrix} \quad \text{and} \quad d = \begin{bmatrix} -2 \\ 2 \end{bmatrix}
\]

we get

\[
cd = 1(-2) + 3(2) = 4
\]

To multiply matrix \( A \) times matrix \( B \), multiply each row of \( A \) times each column of \( B \). The row \( i \) column \( j \) entry of \( AB \) results from multiplying row \( i \) of \( A \) times column \( j \) of \( B \). For example, to get the entry in row 2 and column 3 of \( AB \) multiply row 2 of \( A \) times column 3 of \( B \). Matrix multiplication is defined only if the number of columns of \( A \) is equal to the
number rows of $B$. Thus using $A$ and $B$ from Equation (4.1) we get

$$AB = \begin{bmatrix}
(row 1)(col 1) & (row 1)(col 2) \\
(row 2)(col 1) & (row 2)(col 2)
\end{bmatrix}$$

We mention some important properties of matrix multiplication

- If $A$ is an $i \times j$ matrix and $B$ is a $k \times l$ matrix, then $AB$ is defined if and only if $j = k$ (that is, if the number columns of $A$ is the same as the number of rows of $B$).

- If $A$ is $i \times j$ and $B$ is $j \times k$ then $AB$ is defined and is an $i \times k$ matrix.

- In general $AB \neq BA$ even when both are defined.

**Example 4.1.1** Let $A = \begin{bmatrix} 0 & 2 & -3 \\ 1 & 3 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}$, $c = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $d = -2$. Find $A + B$, $B + c$, $B + B$, $AB$, $BA$, $Ac$, $Bc$, $dA$, and $dc$. If given operation is not defined, say so.

**Solution.** $A + B$ is not defined. $B + c$ is not defined. $B + B = \begin{bmatrix} 2 & -2 \\ 4 & 0 \end{bmatrix}$, $AB$ is not defined.

$$BA = \begin{bmatrix} -1 & -1 & -5 \\ 0 & 4 & -6 \end{bmatrix}.$$ $Ac$ is not defined. $Bc = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$, $dA = \begin{bmatrix} 0 & -4 & 6 \\ -2 & -6 & -4 \end{bmatrix}$, $dc = \begin{bmatrix} -4 \\ 2 \end{bmatrix}$.

**Special matrices, trace, determinant**

Two special matrices of importance are the zero matrix and the identity matrix. The zero matrix is denoted $0$ and is a matrix of all zeros; it can be any size. The size of the matrix can be inferred from context. For example if $A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$ then in the equation $A + 0 = A$ we mean $0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

The identity matrix is a square matrix (same number of rows as columns), which consists of all ones and zeros, in the following arrangement. The elements that appear in row $i$, column $i$, for any $i$, are ones, and all other elements are zeros. Thus, row 2, column 2 has
a one, but row 1, column 2 has a zero. For example the $2 \times 2$ identity is $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and the $3 \times 3$ identity is $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Note that $A + 0 = 0 + A = 0$ and $AI = IA = A$ for any matrix $A$. Thus the zero matrix acts like the number 0 does for real numbers and the identity matrix acts like the number 1 does for real numbers.

The determinant and the trace are defined for square matrices ($n \times n$ matrices of any size).

The determinant of a $2 \times 2$ matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is defined to be $\det(A) = ad - bc$. The determinant is also defined for larger square matrices; see the exercises. For systems larger than $2 \times 2$ we will mostly rely on software to do our calculations for us.

The trace of a square matrix is defined to be the sum of the diagonal elements (the elements in row 1, column 1 and row 2, column 2, and so on). Thus for a $2 \times 2$ matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ we have $\text{trace}(A) = a + d$.

### Systems of algebraic equations in matrix form

Consider the system of algebraic equations

$$
\begin{align*}
2x_1 - x_2 &= 1 \\
x_1 + x_2 &= 3
\end{align*}
$$

(4.2)

This system is linear; also there are two equations and two unknowns. It is systems of this type that we will be interested in - systems of $n$ linear equations with $n$ unknowns. We can put this system into matrix form as follows. Define

$$ A = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 1 \\ 3 \end{bmatrix} $$

Using the definition of matrix multiplication we get $Ax = \begin{bmatrix} 2x_1 - x_2 \\ x_1 + x_2 \end{bmatrix}$. Since two matrices
are equal if and only if all corresponding entries are equal, Equations (4.2) becomes

\[ Ax = b \]

in matrix form.

We will need to use one of the major theorems in linear algebra. It states that the system \( Ax = b \) has either one solution, no solutions or infinitely many solutions for the unknown vector \( x \). Furthermore, if \( A \) is a square matrix, then there is exactly one solution (a unique solution) if and only if \( \det(A) \neq 0 \).

**Example 4.1.2** For each system below, determine whether or not there is a unique solution. If there is a unique solution find it; if not find 3 solutions.

a) \[
\begin{align*}
2x_1 + 3x_2 &= 0 \\
x_1 - x_2 &= 0
\end{align*}
\]

b) \[
\begin{align*}
2x_1 + 4x_2 &= 0 \\
x_1 + 2x_2 &= 0
\end{align*}
\]

**Solution.**

a) We have \( A = \begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix} \) and \( b = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \) so that \( \det(A) = 2(-1) - 1(3) = -5 \) and hence the solution is unique. By inspection, \( \begin{bmatrix} 0 \\ 0 \end{bmatrix} \) (that is, \( x_1 = 0 \) and \( x_2 = 0 \)) is the unique solution. When \( b = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \) the zero vector will always be a solution (do you see why?).

b) We have \( A = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix} \) and \( b = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \) so that \( \det(A) = 2(2) - 1(4) = 0 \) and hence there is not a unique solution. We will again have the solution \( \begin{bmatrix} 0 \\ 0 \end{bmatrix} \).

To find two other solutions, we can solve each of the equations for one of the variables, say \( x_1 \). We get \( x_1 = -2x_2 \) for both the first and second equation. This means we can choose anything we want for one of the variables, and then calculate the other. It also means there are infinitely many solutions. If we choose \( x_2 = 1 \) we get \( x_1 = -2 \) and if we choose \( x_2 = 2 \) we get \( x_1 = -4 \). In vector form the two other solutions are \( \begin{bmatrix} -2 \\ 1 \end{bmatrix} \) and \( \begin{bmatrix} -4 \\ 2 \end{bmatrix} \).
4.1 Matrices, Vectors, Scalars

Systems of DE’s in matrix form

We can also write systems of differential equations in matrix form. First we must define what it means to take the derivative of a vector function. If \( x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} \) then we define

\[
x'(t) = \begin{bmatrix} x'_1(t) \\ x'_2(t) \\ \vdots \\ x'_n(t) \end{bmatrix}, \text{ that is, we differentiate terms by term.}
\]

**Example 4.1.3** Write the DE system

\[
\begin{align*}
x'_1 &= 2x_1 - x_2 \\
x'_2 &= x_1 + x_2
\end{align*}
\]

in matrix form.

**Solution.** Define

\[
A = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}
\]

Then \( AX = \begin{bmatrix} 2x_1 - x_2 \\ x_1 + x_2 \end{bmatrix} \) and \( X' = \begin{bmatrix} x'_1 \\ x'_2 \end{bmatrix} \) so the DE system above becomes

\[
X' = AX
\]

**Exercises**

Given the following matrices

\[
A = \begin{pmatrix} 2 & 1 \\ -3 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 2 \\ 1 & 3 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1 & 2 \\ -3 & -4 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 1 \\ -3 & -4 \end{pmatrix};
\]

Compute each expression 1 - 6 below, if it is defined; otherwise state why it is not defined.

1. \( A + B \)
2. \( 2A + (-3)B \)
3. \( C(A + B) \)
4. \((A + B)C\)
5. \(ABC\)
6. \(A - 2I\)
7. \(2A + 0\)
8. Show that \(AB \neq BA\).
9. Find the trace and determinant of the matrix \(A\).
10. Find the trace and determinant of the matrix \(B\).

For each of the following algebraic systems of equations, identify the matrix \(A\) and the vector \(b\) that corresponds to the matrix equation \(Ax = b\).

11. \(x_1 - x_2 = 2, \ 3x_1 - 4x_2 = 5\)
12. \(-x_1 - 3x_2 = 0, \ -2x_2 = 4\)

For each of the following systems of differential equations, identify the matrix \(A\) that corresponds to the matrix equation \(x' = Ax\).

13. \(x'_1 = x_2, \ x'_2 = x_1 - x_2\)
14. \(x'_1 = -x_1 + 3x_2, \ x'_2 = 2x_1 - 4x_2\)

## 4.2 Eigenvalues and Eigenvectors

Given a square matrix \(A\) (for this text, we mostly work with \(2 \times 2\) matrices), the eigenvalue-eigenvector problem is to find scalars \(r\) and vectors \(u\) (we want non-zero \(u\)) for which

\[
Au = ru \tag{4.3}
\]

The scalars are the eigenvalues and the vectors are the eigenvectors. Using a little matrix algebra Equation (4.3) can be written as \(Au = rIu\) and then \(Au - ru = 0\) and finally

\[
(A - rI)u = 0 \tag{4.4}
\]

We will call equation Equation (4.4) the eigenvector equation.
4.2 Eigenvectors and Eigenvectors

One vector \( u \) that always solves Equation (4.4) is the zero vector \( u = 0 \). From the previous section we know that Equation (4.4) must have either one solution, no solutions, or infinitely many solutions. Thus for there to be any non-zero \( u \) that are solutions, there must be infinitely many such solutions, and again from the previous section, this requires that

\[
\det(A - rI) = 0
\]  

(4.5)

We will call Equation (4.5) the eigenvalue equation. For each eigenvalue \( r \) that you get from Equation (4.5), use Equation (4.4) \((A - rI)u = 0\) to get a corresponding eigenvector \( u \). You should end up with pairs \((r_1, u_1), (r_2, u_2)\) and so on (for \( 2 \times 2 \) matrices you get two pairs).

NOTE: If \( r_1 = r_2 \) we have a repeated eigenvalue. In this case there can be either one or two (linearly independent) eigenvectors.

**Example 4.2.1** Let \( A = \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix} \). Find the eigenvalue/eigenvector pairs.

**Solution.** The Equation (4.5), the eigenvalue equation \( \det(A - rI) = 0 \),

becomes \( \det \left( \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix} - r \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) = \det \left( \begin{bmatrix} 1 - r & 4 \\ 1 & 1 - r \end{bmatrix} \right) \)

\( = (1 - r)^2 - 4 = 0 \). The solutions to this quadratic are \( r_1 = -1 \) and \( r_2 = 3 \); they are the eigenvalues.

For the eigenvectors we use Equation (4.4), the eigenvector equation \((A - rI)U = 0\).

By letting \( U = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \), this becomes \( \begin{bmatrix} 1 & -4 \\ 1 & 1 \end{bmatrix} - r \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \) or

\[
\begin{bmatrix} 1 - r & 4 \\ 1 & 1 - r \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]

This results in two equations \((1 - r)x_1 + 4x_2 = 0\) and \( x_1 + (1 - r)x_2 = 0 \). For the eigenvalue \( r_1 = -1 \) these equations become \( 2x_1 + 4x_2 = 0 \) and \( x_1 + 2x_2 = 0 \); both equations give \( x_1 = -2x_2 \). This means that we only need one of the two equations, and we can choose any value we want for \( x_2 \); for example if \( x_2 = 1 \) then \( x_1 = -2x_2 = -2 \) and the eigenvector is \( U_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \).
For the eigenvalue $r_1 = 3$ the eigenvector equations $(1 - r)x_1 + 4x_2 = 0$ and $x_1 + (1 - r)x_2 = 0$ become $-2x_1 + 4x_2 = 0$ and $x_1 - 2x_2 = 0$; both equations give $x_1 = 2x_2$. Again we only need one of the two equations, and we can choose any value we want for $x_2$; if $x_2 = 1$ then $x_1 = 2x_2 = 2$ and the eigenvector is $U_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

We now have our two eigenvalue-eigenvector pairs. They are $(-1, \begin{bmatrix} -2 \\ 1 \end{bmatrix})$ and $(3, \begin{bmatrix} 2 \\ 1 \end{bmatrix})$.

Properties of eigenvalues and eigenvectors.

1. Any multiple of an eigenvector is the same eigenvector. For example $x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is the same eigenvector as $x = \begin{bmatrix} 0.44721 \\ 0.89443 \end{bmatrix}$. This is because if $Au = \lambda u$ then $A(cu) = \lambda(cu)$ (multiply both sides by the scalar $c$) which shows that $cu$ is an eigenvector. The second form above is a unit vector, and is how the TI gives you eigenvectors.

2. Eigenvalues and eigenvectors can be real or complex. There are three cases: two real eigenvalues, one real eigenvalue, two complex eigenvalues (sound familiar?). On the TI89 make sure that under MODE the Complex Format is set to Rectangular.

3. When the eigenvalues are complex they come in complex conjugate pairs $r = \alpha \pm \beta$. When this happens, the eigenvectors also come in complex conjugate pairs $U = V \pm iW$. See the next example.

Eigenvalues and eigenvectors on the TI89

- To store a matrix (such as $A = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}$) in a variable (such as $a$) use $[1, 2; 3, -1] \rightarrow a$.

- Use math-matrix-eigVl for eigenvalues and math-matrix-eigVc for eigenvectors.

- The first column of the eigenvector matrix is the eigenvector that corresponds to the first eigenvalue (second column corresponds to second eigenvalue).
Example 4.2.2  Let \( A = \begin{bmatrix} 1 & -4 \\ 1 & 1 \end{bmatrix} \). The eigenvalues and eigenvectors are complex; find \( \alpha, \beta, V, \) and \( W \).

Solution.  After storing the matrix and giving the eigenvalue command, the TI89 gives us \( r_1 = 1 + 2i \) and \( r_2 = 1 - 2i \). Thus \( \alpha = 1 \) and \( \beta = 2 \). The eigenvector command gives 
\[
\begin{bmatrix}
0.894427191 & 0.894427191 \\
-0.4472135955i & 0.4472135955i
\end{bmatrix}.
\]
The first eigenvector is 
\[
\begin{bmatrix}
0.894427191 \\
0.4472135955i
\end{bmatrix},
\]
which can be expanded into its real and imaginary parts as 
\[
\begin{bmatrix}
0.894427191 \\
0
\end{bmatrix} + \begin{bmatrix}
0 \\
-0.4472135955i
\end{bmatrix} i. \]

This means that 
\[
V = \begin{bmatrix}
0.894427191 \\
0
\end{bmatrix} \quad \text{and} \quad W = \begin{bmatrix}
0 \\
-0.4472135955
\end{bmatrix}.
\]

The result can be simplified by multiplying the complex eigenvector 
\[
\begin{bmatrix}
0.894427191 \\
-0.4472135955i
\end{bmatrix}
\]
by \( \frac{1}{0.4472135955} \) to get the complex eigenvector 
\[
\begin{bmatrix}
2 \\
-1
\end{bmatrix}
\]
which expands to 
\[
\begin{bmatrix}
2 \\
0
\end{bmatrix} + \begin{bmatrix}
0 \\
1
\end{bmatrix} i. \]

Then one can use 
\[
V = \begin{bmatrix}
2 \\
0
\end{bmatrix} \quad \text{and} \quad W = \begin{bmatrix}
0 \\
-1
\end{bmatrix}.
\]

Three things to note: 1) both \( V \) and \( W \) are real vectors, 2) neither \( V \) nor \( W \) are eigenvectors, and 3) only the first complex eigenvector is needed (the second one would lead to the same \( V \) and \( W \)).

Exercises
For each of the matrices 1 - 6 below, find the eigenvalues. Find an eigenvector corresponding to each eigenvalue. In each case, do this first by hand and then use technology (TI-86, TI-89, Maple, etc.). Explain any differences.

1. \( A = \begin{bmatrix} -2 & 2 \\ 1 & -3 \end{bmatrix} \)

2. \( B = \begin{bmatrix} 2 & 1 \\ 0 & -3 \end{bmatrix} \)

3. \( C = \begin{bmatrix} -4 & 2 \\ -2 & 0 \end{bmatrix} \)
4. \( D = \begin{pmatrix} 1 & -4 \\ 4 & -7 \end{pmatrix} \)

5. \( E = \begin{pmatrix} 1 & 3 \\ -3 & 1 \end{pmatrix} \)

6. \( F = \begin{pmatrix} 2 & 8 \\ -1 & -2 \end{pmatrix} \)
Chapter 5

Systems of First-order Differential Equations

5.1 Explicit Solutions of Constant-coefficient Linear Systems

From the second linear algebra section we know that a system of DE’s \( x_1' = ax_1 + bx_2 \), \( y' = cx_1 + dx_2 \) can be written in matrix form as

\[
\begin{bmatrix}
  x_1 \\
  x_2
\end{bmatrix}' = \begin{bmatrix}
  a & b \\
  c & d
\end{bmatrix} \begin{bmatrix}
  x_1 \\
  x_2
\end{bmatrix} \quad \text{or} \quad X' = AX
\]

where \( X = \begin{bmatrix}
  x_1 \\
  x_2
\end{bmatrix} \) and \( A = \begin{bmatrix}
  a & b \\
  c & d
\end{bmatrix} \). To solve \( X' = AX \) we assume a solution of the form \( X(t) = e^{rt}U \) where \( U \) is a vector of constants \( U = \begin{bmatrix}
  u_1 \\
  u_2
\end{bmatrix} \). This leads to \( X' = re^{rt}U \) so that the equation \( X' = AX \) becomes \( re^{rt}U = Ae^{rt}U \). We then divide out the \( e^{rt} \) and rearrange to get \( AU = rU \) which is the eigenvalue-eigenvector equation, Equation (4.3).

The General Solution

Recall that there are three cases for the eigenvalues: two real eigenvalues, two complex eigenvalues, and one real eigenvalue (also referred to as repeated eigenvalues). If we have two distinct eigenvalues, then we get two (linearly independent) solutions \( X_1 = e^{rt_1}U_1 \) and
\[ X_2 = e^{r_2t}U_2. \] We can then take a linear combination \( C_1X_1 + C_1X_2 \) for the general solution, similar to when we solved second order linear equations.

Thus the general solution to \( X' = AX \) for the 2 real eigenvalues case is

\[
X = C_1e^{r_1t}U_1 + C_2e^{r_2t}U_2 \quad \text{(Two real eigenvalues)}
\]

For the case of two complex eigenvalues, we again proceed in a manner similar to what we did in Chapter 3 for second order equations, and invoke the property of linear differential equations (or systems) that the real and imaginary parts of a complex solution to a DE or DE system are real valued solutions. Given that \( e^{rt}u \) is a solution, write \( r = \alpha + \beta i \) and \( u = V + Wi \) to get \( e^{rt}u = e^{\alpha t}(\cos(\beta t) + i \sin(\beta t))(V + Wi) = e^{\alpha t}(\cos(\beta t)V - \sin(\beta t)W + \\
\cos(\beta t)W + \sin(\beta t)V)i \) so that the real and imaginary parts are \( e^{\alpha t}(\cos(\beta t)V - \sin(\beta t)W) \) and \( e^{\alpha t}(\cos(\beta t)V + \sin(\beta t)W) \).

Thus the general solution to \( X' = AX \) for the 2 complex eigenvalues case is

\[
X = C_1e^{\alpha t}(\cos(\beta t)V - \sin(\beta t)W) + C_2e^{\alpha t}(\cos(\beta t)V + \sin(\beta t)W) \quad \text{(Two complex eigenvalues)}
\]

where the eigenvalues are \( \alpha \pm \beta i \) and the eigenvectors are \( V \pm Wi \).

**Example 5.1.1** Find the general solutions to each system of differential equations.

a) \( x' = x + 2y, \ y' = 8x + y \)

b) \( x' = x + 2y, \ y' = -8x + y \)

**Solution.**

a) We have \( A = \begin{bmatrix} 1 & 2 \\ 8 & 1 \end{bmatrix} \). The eigenvalues and corresponding eigenvectors (from the TI89) are \( r_1 = 5, \ r_2 = -3 \) and \( U_1 = \begin{bmatrix} 0.447214 \\ 0.894427 \end{bmatrix} \), and \( U_2 = \begin{bmatrix} -0.447214 \\ 0.894427 \end{bmatrix} \). Since a multiple of
an eigenvector is still an eigenvector, if we multiply each eigenvector by $\frac{1}{0.447214}$ we get $U_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $U_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ which provide for a “nicer” solution. The general solution is now

$$X = C_1 e^{5t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + C_2 e^{-3t} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} C_1 e^{5t} - C_2 e^{-3t} \\ 2C_1 e^{5t} + 2C_2 e^{-3t} \end{bmatrix}$$

In component form (using the original variables) we get

$$x_1 = C_1 e^{5t} - C_2 e^{-3t} \text{ and } x_2 = 2C_1 e^{5t} + 2C_2 e^{-3t}$$

b) We have $A = \begin{bmatrix} 1 & 2 \\ -8 & 1 \end{bmatrix}$. The eigenvalues and corresponding eigenvectors (from the TI89) are $r_1 = 1 + 4i$, $r_2 = 1 - 4i$ and $U_1 = \begin{bmatrix} -0.447214i \\ 0.894427 \end{bmatrix}$, and $U_2 = \begin{bmatrix} 0.447214i \\ 0.894427 \end{bmatrix}$. Since a multiple of an eigenvector is still an eigenvector, if we multiply each eigenvector by $\frac{1}{0.447214}$ we get $U_1 = \begin{bmatrix} -i \\ 2 \end{bmatrix}$ and $U_2 = \begin{bmatrix} i \\ 2 \end{bmatrix}$. Expanding the complex eigenvectors into their real and imaginary parts we get $U_1 = \begin{bmatrix} 0 - 1i \\ 2 + 0i \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \end{bmatrix} i$ and $U_2 = \begin{bmatrix} 0 + 1i \\ 2 + 0i \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix} i$.

From the eigenvalues we have $\alpha = 1$ and $\beta = 4$, and from the eigenvectors we have $V = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$ and $W = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$. Notice that we really only needed the first eigenvalue and first eigenvector. Using the general solution from the two complex eigenvalue case we get

$$X = C_1 e^{t} \begin{bmatrix} \cos(4t) \\ 2 \end{bmatrix} - \sin(4t) \begin{bmatrix} -1 \\ 0 \end{bmatrix} + C_2 e^{t} \begin{bmatrix} \cos(4t) \\ 2 \end{bmatrix} - \sin(4t) \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

We could also write this solution as

$$X = \begin{bmatrix} C_1 e^{t} \sin(4t) - C_2 e^{t} \cos(4t) \\ 2C_1 e^{t} \cos(4t) + 2C_1 e^{t} \sin(4t) \end{bmatrix}$$

so that in component form we have

$$x_1 = C_1 e^{t} \sin(4t) - C_2 e^{t} \cos(4t) \text{ and } x_2 = 2C_1 e^{t} \cos(4t) + 2C_1 e^{t} \sin(4t)$$
For the case of one real eigenvalue (repeated eigenvalues) we need to come up with a second linearly independent solution. We will not cover this case at this time.

Initial Conditions

We treat systems with initial conditions the same way we did first-order and second-order equations: after the general solution is found, use the initial conditions to find the constants in the general solution. We illustrate with an applied example.

Example 5.1.2 In Hawaii, feral chickens have taken over the island of Kauai (fact). Let us assume that in one particular field, the chickens only source of food is worms, and at the moment the chickens and worm populations are stable at a level of 30 chickens and 2000 worms (we call these equilibrium population values). Now let $x$ = the number of chickens above equilibrium, and let $y$ = the number of worms above equilibrium. Thus if $x = 5$ and $y = -100$ that means there are 35 chickens and 1900 worms. Though the model for a predator-prey situation such as this is actually nonlinear, as long as the chicken and worm populations are near the equilibrium values, the following linear model works well

$$x' = -0.12x - 12y, \quad y' = 0.006x$$

where time is measured in months. If $x$ and $y$ both start at zero, they stay at zero (recall this means 30 chickens and 2000 worms). Do you see why?

Now assume that 5 chickens are kidnapped (chickennapped?). What will happen to both populations in the future? Use the initial conditions $x = -5$ and $y = 0$ (corresponding to 25 chickens and 2000 worms). Predict the size of both populations 3 months later, and 10 years later.

Solution. We can find the general solution to the system above, then use the initial conditions to determine the constants, and finally predict the future using $t = 3$ and $t = 120$.

The system can be written as $X' = AX$, where $A = \begin{bmatrix} -0.12 & -12 \\ 0 & 0.006 \end{bmatrix}$ and $X = \begin{bmatrix} x \\ y \end{bmatrix}$.

The eigenvalues are $-0.06 \pm 0.26153i$ and the eigenvectors are $\begin{bmatrix} -0.99975 \\ 0.005 \end{bmatrix} \pm \begin{bmatrix} 0 \\ 0.021789 \end{bmatrix} i$. Thus we are in the complex case with $\alpha = -0.06$, $\beta = 0.26153$, $\gamma = 0.006$, and $\omega = 0.26153$. 

$$X = \begin{bmatrix} x \\ y \end{bmatrix} = C_1 \begin{bmatrix} -0.99975 \\ 0.005 \end{bmatrix} e^{-0.06t} \cos(0.26153t) + C_2 \begin{bmatrix} 0.99975 \\ -0.005 \end{bmatrix} e^{-0.06t} \sin(0.26153t)$$

For $t = 3$ months, $X = \begin{bmatrix} 25 \\ 2000 \end{bmatrix}$.

For $t = 120$ years, $X = \begin{bmatrix} 30 \\ 2000 \end{bmatrix}$.
Explicit Solutions of Constant-coefficient Linear Systems

\[ V = \begin{bmatrix} -0.99975 \\ 0.005 \end{bmatrix}, \text{ and } W = \begin{bmatrix} 0 \\ 0.021789 \end{bmatrix}. \]

The general solution, using the complex case equation (and rounding a bit), is

\[ X = C_1 e^{-0.06t} \begin{bmatrix} 0 \\ 0.021789 \end{bmatrix} - \sin(0.262t) \begin{bmatrix} -0.99975 \\ 0.005 \end{bmatrix} + \sin(0.262t) \begin{bmatrix} 0 \\ 0.021789 \end{bmatrix} \]
\[ + C_2 e^{-0.06t} \begin{bmatrix} 0 \\ 0.021789 \end{bmatrix} + \sin(0.262t) \begin{bmatrix} -0.99975 \\ 0.005 \end{bmatrix} \tag{5.1} \]

We now use the initial conditions \( x(0) = 25 \) and \( y(0) = 2000 \), or in vector form \( \begin{bmatrix} -5 \\ 0 \end{bmatrix} \). From Equation (5.1) we have

\[ X(0) = C_1 \begin{bmatrix} -0.99975 \\ 0.005 \end{bmatrix} + C_2 \begin{bmatrix} 0 \\ 0.021789 \end{bmatrix} = \]
\[ \begin{bmatrix} -0.99975C_1 \\ 0.005C_1 + 0.021789C_2 \end{bmatrix}. \]

Setting the two expressions for \( X(0) \) equal, we get the two equations

\[-0.99975C_1 = -5 \text{ and } 0.005C_1 + 0.021789C_2 = 0\]

The solution to these equations is \( C_1 = 5.0013 \), \( C_2 = -1.1477 \). Substituting into Equation (5.1) we get

\[ X = 5.0013 e^{-0.06t} \begin{bmatrix} 0 \\ 0.021789 \end{bmatrix} - \sin(0.262t) \begin{bmatrix} -0.99975 \\ 0.005 \end{bmatrix} + \sin(0.262t) \begin{bmatrix} -0.99975 \\ 0.005 \end{bmatrix} \]
\[ -1.1477 e^{-0.06t} \begin{bmatrix} 0 \\ 0.021789 \end{bmatrix} + \sin(0.262t) \begin{bmatrix} -0.99975 \\ 0.005 \end{bmatrix} \]

which upon multiplying out (and rounding) gives the component equations

\[ x = -5.0 (\cos 0.2615t) e^{-0.06t} + 1.1471 (\sin 0.2615t) e^{-0.06t} \]
\[ y = -0.1147 (\sin 0.2615t) e^{-0.06t} \]

Finally, substituting \( t = 3 \) and \( t = 120 \) into the above equations we get

\[ x(3) = -2.279 \quad \text{and} \quad x(120) = -0.004 \]
\[ y(3) = -0.068 \quad \text{and} \quad y(120) = 0.000 \]

accurate to three decimal places. However, since both \( x \) and \( y \) represent the size of a
population, we conclude that after 3 months $x = -2$ and $y = 0$, which means $30 - 2 = 28$ chickens and 2000 worms. After 10 years we are back to 30 chickens and 2000 worms. Thus it appears that if the number of chickens and worms is varied slightly from the equilibrium values of 30 chickens and 2000 worms, the two populations eventually return to the equilibrium values.

**Exercises** Find solutions to each of the following systems, given in component form. If initial conditions are not given, find a general solution. Give the solution in both vector form and component form (that is, give expressions for $x(t)$ and $y(t)$).

1. $x' = y$, $y' = x$.
2. $x' = -y$, $y' = x$.
3. $x' = 2x - y$, $y' = y$.
4. $x' = -x$, $y' = x$, $x(0) = 1$, $y(0) = -1$.
5. $x' = x + y$, $y' = -x + y$, $x(0) = 0$, $y(0) = 1$.
7. Solve problem 5 again, without using matrices.

---

### 5.2 Stability of Autonomous Linear Systems

**Fixed Points**

A fixed point (equilibrium point, constant solution) of a system $x' = f(x,y)$, $y' = g(x,y)$ is a point $(x, y)$ for which $f(x,y) = 0$ and $g(x,y) = 0$. For a linear, constant coefficient, autonomous system $x' = ax + by$, $y' = cx + dy$, the only fixed point is $(0,0)$ as long as $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} \neq 0$ (if the determinant is zero there are infinitely many fixed points, including $(0,0)$). Thus if the initial condition of such a system is the origin $(0,0)$ the solution stays at $(0,0)$ for all time.
Stability

If every solution curve of a linear system that starts near \((0, 0)\) stays near \((0, 0)\), then \((0, 0)\) is called *stable*. Otherwise it is unstable.

The Case of Two Real Eigenvalues

Recall that the algebraic solution is given by \(X = C_1 e^{r_1 t} U_1 + C_2 e^{r_2 t} U_2\) for the case of two real eigenvalues. If one of the constants \(C_i\) is zero, then the solution is given by a scalar multiple of an eigenvector (the term with the nonzero constant). Since a scalar multiple of an eigenvector is still the same eigenvector, this means that the eigenvectors themselves are straight line solutions. If your initial condition lies on an eigenvector, the solution stays on the eigenvector (because the \(C_i\) for the other term will be zero).

Now recall that \(e^{rt} \to 0\) as \(t \to \infty\) if \(r\) is negative, and \(e^{rt} \to \infty\) as \(t \to \infty\) if \(r\) is positive. Putting this together with the previous paragraph, we can conclude that

1. If the initial condition of a solution curve is on an eigenvector \(U_i\) for which the corresponding eigenvalue \(r_i\) is positive, the solution will stay on \(U_i\) and move away from the origin \((0, 0)\) (both components of the solution will approach \(\pm \infty\)). We can call this an *outgoing* eigenvector.

2. If the initial condition of a solution curve is on an eigenvector \(U_i\) for which the corresponding eigenvalue \(r_i\) is negative, the solution will stay on \(U_i\) and move towards the origin \((0, 0)\) (both components of the solution will approach zero). We can call this an *incoming* eigenvector.

3. If both \(r\)'s are positive, the solution is unstable because both eigenvectors are outgoing. This is called a *source*.

4. If one \(r\) is positive and one is negative, the solution is unstable because one of the eigenvectors is outgoing. This is called a *saddle*. Note: if the initial condition is on the incoming eigenvector, the solution will go to \((0, 0)\), but this will not be true for *every* initial condition near \((0, 0)\).

5. If both \(r\)'s are negative, the solution is stable because both eigenvectors are incoming. This a called a *sink*.

6. Though we did not solve the case algebraically where \(r_1 = r_2\), we will simply state the result that if \(r_1 = r_2 > 0\) we have a source (unstable) and if \(r_1 = r_2 < 0\) we have a sink (stable).
Two Complex Eigenvalues

Recall that the algebraic solution is given by \( X = C_1 e^{\alpha t}(\cos(\beta t)V - \sin(\beta t)W) + C_2 e^{\alpha t}(\cos(\beta t)W + \sin(\beta t)V) \) for the case of two complex eigenvalues \( r = \alpha \pm \beta i \). For this case the eigenvectors are not real-valued solutions (they are complex also). However, the stability analysis is actually simpler, in that both terms in \( X \) are multiplied by \( e^{\alpha t} \), and so \( \alpha \) completely determines the stability.

1. If \( \alpha > 0 \) the solution is unstable. Solution curves oscillate with increasing amplitude. This is called a *spiral source*.

2. If \( \alpha < 0 \) the solution is stable. Solution curves oscillate with decreasing amplitude and approach the origin. This is called a *spiral sink*.

3. If \( \alpha = 0 \) the solution is stable. Solution curves oscillate with constant amplitude. This is called a *center*. It is considered stable because, even though solution curves do not approach \((0,0)\), they can be kept close to the origin by starting close to it.

See Figure 5.1 for graphs of the six basic stability types.
5.2 Stability of Autonomous Linear Systems

Zero Eigenvalues

If one of the eigenvalues is zero (a real number), and the other non-zero, the second one must also be a real number. In this case, there will be two real eigenvectors, but the eigenvector corresponding to the zero eigenvalue will consist of all fixed points (if you start on this eigenvector you stay put). Stability is determined by the non-zero eigenvalue; if it is positive the origin is unstable, and if it is negative the origin is stable (though solution curves do not necessarily approach the origin).

If both eigenvalues are zero, all the points in the plane are fixed points, are are stable, including the origin (not a very interesting case).

**Example 5.2.1** For each system, determine the stability type of $(0,0)$. 

![Stability Types](image-url)
1. \( x' = x + y, \quad y' = 4x + y \)
2. \( x' = x - y, \quad y' = 4x + y \)
3. \( x' = x, \quad y' = x + 3y \)
4. \( x' = y, \quad y' = -x \)

Solution.

1. \( A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \). The eigenvalues are \(-1\) and \(3\), so the origin is a saddle.

2. \( A = \begin{bmatrix} 1 & -1 \\ 4 & 1 \end{bmatrix} \). The eigenvalues are \(1 \pm 2i\), so the origin is a spiral source.

3. \( A = \begin{bmatrix} 1 & 0 \\ 1 & 3 \end{bmatrix} \). The eigenvalues are \(1\) and \(3\), so the origin is a source.

4. \( A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \). The eigenvalues are \(\pm i\) so the origin is a center.

The Trace-Determinant Plane

When the system is written in matrix form \( x' = Ax \), the trace and determinant of \( A \) can also be used to determine the stability type of \((0, 0)\). Recall that the determinant of \( A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \) is \(ad - bc\) and the trace is \(a + d\).

A theorem in linear algebra (not stated previously) states that the determinant of a matrix is the product of the eigenvalues, and the trace is the sum of the eigenvalues. Thus \( r_1 + r_2 = \text{trace}(A) \) and \( r_1r_2 = \det(A) \). If we solve these two equations for \( r_1 \) and \( r_2 \) we get

\[
r_1, r_2 = \frac{1}{2} tr \pm \frac{1}{2} \sqrt{tr^2 - 4det} = \frac{1}{2} tr \pm \sqrt{\frac{tr^2}{4} - det}
\]

From this equation we can conclude (proof relegated to the exercises) that

1. If \( \det < 0 \), the eigenvalues are real and opposite sign; this means \((0, 0)\) is a saddle.
2. If \(0 < \det < \frac{tr^2}{4}\) the eigenvalues are real; if \(tr > 0\) they are both positive (source) and if \(tr < 0\) they are both negative (sink).

3. If \(\det > \frac{tr^2}{4}\) the eigenvalues are complex; if \(tr > 0\) the real part \(\alpha > 0\) (spiral source), if \(tr < 0\) the real part \(\alpha < 0\) (spiral source), and if \(tr = 0\) the real part \(\alpha = 0\) (center).

The table below summarizes the information in this section, by showing how the trace and determinant of \(A\) and/or the eigenvalues of \(A\) determine the stability type of \((0, 0)\) for \(x' = Ax\)

<table>
<thead>
<tr>
<th>Stability type</th>
<th>trace/determinant</th>
<th>eigenvalues</th>
</tr>
</thead>
<tbody>
<tr>
<td>saddle</td>
<td>(\det &lt; 0)</td>
<td>one (\lambda &gt; 0) and one (\lambda &lt; 0)</td>
</tr>
<tr>
<td>sink</td>
<td>(\det &gt; 0, \text{trace} &lt; 0, \det \leq \frac{1}{4}\text{trace}^2)</td>
<td>(\lambda's) real and negative</td>
</tr>
<tr>
<td>spiral sink</td>
<td>(\det &gt; 0, \text{trace} &lt; 0, \det &gt; \frac{1}{4}\text{trace}^2)</td>
<td>(\lambda's) complex, (\alpha &lt; 0)</td>
</tr>
<tr>
<td>center</td>
<td>(\det &gt; 0, \text{trace} = 0)</td>
<td>(\alpha = 0)</td>
</tr>
<tr>
<td>spiral source</td>
<td>(\det &gt; 0, \text{trace} &gt; 0, \det &gt; \frac{1}{4}\text{trace}^2)</td>
<td>(\lambda's) complex, (\alpha &gt; 0)</td>
</tr>
<tr>
<td>source</td>
<td>(\det &gt; 0, \text{trace} &gt; 0, \det \leq \frac{1}{4}\text{trace}^2)</td>
<td>(\lambda's) real and positive</td>
</tr>
</tbody>
</table>

The trace and determinant part of the table above can be understood more easily with a graph. See Figure 5.2. The trace is plotted on the \(x - axis\) (horizontal axis) and the determinant is plotted on the \(y - axis\) (vertical axis). Just plot the trace and determinant as an ordered pair \((tr, det)\) and see which region it falls in. The positive vertical axis corresponds to the stability type center.

![Figure 5.2: Stability and the trace determinant plane](image)
Example 5.2.2 For each system, determine the stability type of \((0, 0)\) using the trace and determinant. These are the same equations from the previous example.

1. \(x' = x + y, \ y' = 4x + y\)
2. \(x' = x - y, \ y' = 4x + y\)
3. \(x' = x, \ y' = x + 3y\)
4. \(x' = y, \ y' = -x\)

Solution.

1. \(A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}\). The determinant is \(-3\). Since \(det < 0\) the origin is a saddle.

2. \(A = \begin{bmatrix} 1 & -1 \\ 4 & 1 \end{bmatrix}\). The trace is 2 and the determinant is 5. Since \(\frac{1}{4} tr^2 = 1\) we have \(det > \frac{1}{4} tr^2\) so the origin is a spiral source.

3. \(A = \begin{bmatrix} 1 & 0 \\ 1 & 3 \end{bmatrix}\). The trace is 4 and the determinant is 3. Since \(\frac{1}{4} tr^2 = 4\) we have \(det < \frac{1}{4} tr^2\) so the origin is a source.

4. \(A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\). The trace is 0 and the determinant is 1. Thus the origin is a center.

Exercises For each matrix \(A\) below, compute the trace and determinant. Use the trace-determinant plane to decide what type of equilibrium the system \(\mathbf{X}' = A\mathbf{X}\) has at \((0, 0)\).

1. \(A = \begin{bmatrix} 1 & 2 \\ -2 & -2 \end{bmatrix}\)
2. \(A = \begin{bmatrix} 2 & 3 \\ -1 & 0 \end{bmatrix}\)
For each system below, find the eigenpairs (you will probably want to use technology for this) and sketch by hand a phase portrait for the system. If the eigenvalues are real, the eigenvectors should be included in the sketch. Put arrows on each solution trajectory to denote the direction of motion.

7. \( x' = x + 2y, \ y' = -2x - 2y \)
8. \( x' = 2x + 3y, \ y' = -x \)
9. \( x' = -4x + 2y, \ y' = -x + y \)
10. \( x' = -x + y, \ y' = x - 2y \)
11. \( x' = 2x + 3y, \ y' = -2x - 2y \)
12. \( x' = x + 2y, \ y' = x + 3y \)

For each mass-spring equation 13-18 below, write it as a system and use the trace-determinant plane or eigenvalues to determine the type of equilibrium at \((0,0)\). How does the type of the equilibrium compare with the type of damping (undamped, under damped, critically damped, or over damped)? Can you formulate a general statement about this?

13. \( x'' + 4x' + 2x = 0 \)
14. \( x'' + 9x = 0 \)
15. \( 3x'' + 2x' + x = 0 \)
16. \( x'' + x' + 2x = 0 \)
17. \( x'' + 5x' + 4x = 0 \)
18. \( x'' + 2x' + x = 0 \)
5.3 Fixed Points and Stability of Nonlinear Autonomous Systems

Nonlinear autonomous systems are of the form \( x' = f(x, y), \ y' = g(x, y) \). The fixed points (constant solutions) are the ordered pairs \((x_1, y_1), (x_2, y_2), \ldots\) for which \( f(x, y) = 0 \) and \( g(x, y) = 0 \). We can solve the equations \( f(x, y) = 0 \) and \( g(x, y) = 0 \) using the TI89 with the Solve command (use the word ‘and’ between the equations). There can be many fixed points for the nonlinear case, and it can be difficult to find all of them without software (and even then you may not find them all).

In order to find the stability type of a fixed point, we can create a linear system (this is called linearizing the nonlinear system) for which the origin of the linear system has basically the same stability type as the fixed point of the nonlinear system. We can do this for each fixed point of the nonlinear system. The key to doing this is finding the Jacobian matrix.

The Jacobian matrix is defined as 
\[
J = \begin{bmatrix}
\frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\
\frac{\partial g}{\partial x} & \frac{\partial g}{\partial y}
\end{bmatrix}
\].

We then evaluate the Jacobian at each fixed point to get as many matrices \( J(x_1, y_1), J(x_2, y_2), \ldots \) as there are fixed points. Finally, we can use the trace/det plane or the eigenvalues of the Jacobian matrices to determine stability of each fixed point.

There is one important exception when using the Jacobian to determine stability. If the stability type of the linearized system (corresponding to the Jacobian) is a center, the actual stability type of the nonlinear system could be spiral sink, spiral source or center. This is the only case that is inconclusive.

After the stability type of each fixed point is determined, a phase portrait can be sketched, using both software and the knowledge of the stability of each fixed point. The fixed points are crucial in knowing what viewing window to use with your software.

**Example** 5.3.1 For the system \( x' = x + y, \ y' = y^3 - y \)

1. Find all fixed points.

2. Determine the stability type of each fixed point using the Jacobian.

3. Sketch an accurate phase portrait using the stability information you found. You may also want to use software (such as TI89 or an applet) to help with the sketch. The sketch should clearly show the stability type of each fixed point.
Solution. Solving the system $x + y = 0$, $y^3 - y = 0$ we get the ordered pairs $(1, -1)$, $(-1, 1)$, and $(0, 0)$.

The Jacobian matrix is $J = \begin{bmatrix} 1 & 1 \\ 0 & 3y^2 - 1 \end{bmatrix}$. At the fixed points $(1, -1)$ and $(-1, 1)$ we get $J(1, -1) = J(-1, 1) = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$. With trace $= 3$, $\frac{1}{4} trace^2 = \frac{9}{4}$, and determinant $= 2$ we have determinant $< \frac{1}{4} trace^2$ so both of these fixed points are sources. At $(0, 0)$ we get $J(0, 0) = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$. With determinant $= -1$ we have a saddle. The sketch is shown in Figure 5.3.

![Phase portrait for $x' = x + y$, $y' = y^3 - y$](image)

**Figure 5.3:** Phase portrait for $x' = x + y$, $y' = y^3 - y$

**Exercises**

For each system below:

a. Find all of the equilibrium points in the given region of the phase plane. If a region is not given, choose an appropriate one based on the equilibrium points.

b. Determine the stability type of each equilibrium point using the Jacobian. You may use
eigenvalues or trace/determinant.

c. Use the information from parts a-c to sketch a phase portrait; you may also want to use software to help.

1. \( x' = y, \ y' = -0.5y + x - x^3 \)
2. \( x' = y, \ y' = -0.5y - x + x^3 \)
3. \( x' = y, \ y' = -0.5y + x - x^2 \)
4. \( x' = y, \ y' = -0.5y - x + x^2 \)

5. **Competing species model.** The functions \( x(t) \) and \( y(t) \) represent the population size, at time \( t \), of two competing species in the same ecosystem. Their growth equations are given by

\[
\begin{align*}
x' &= x(1 - x) - xy \\
y' &= y\left(\frac{3}{4} - y\right) - \frac{1}{2}xy.
\end{align*}
\]

Let \( x \) vary between 0 and 1.5, and \( y \) between 0 and 1.5.

6. **Damped pendulum model.** The second-order equation \( \theta'' + \theta' + 4\sin(\theta) = 0 \) is converted into the system

\[
\begin{align*}
x' &= y \\
y' &= -4\sin(x) - y.
\end{align*}
\]

Let \( x \) vary between \(-8\) and \(8\), and \( y \) between \(-10\) and \(10\).

7. **Predator-prey model.** In this model, \( x(t) \) represents the number of predators at time \( t \), and \( y(t) \) is the number of prey. Notice that the predators are affected positively by their interactions with the prey, while the affect of the interaction on the prey is negative. If no prey are available, the predators will die out exponentially.

\[
\begin{align*}
x' &= -x + xy \\
y' &= 4y - 2xy.
\end{align*}
\]

Let \( x \) vary between 0 and 5, and \( y \) between 0 and 5.
5.4 Bifurcations in Systems

A second-order differential equation or system of first-order equations may have an unspecified parameter in it. In this case we can still often determine the stability of the fixed points using the trace and determinant or the eigenvalues of the Jacobian. Since the trace and determinant often result in simpler expressions to work with than the eigenvalues, we will focus on the former approach in this section.

**Example 5.4.1** For the system of equations given by

\[
\begin{align*}
x' &= -x(1-x) + y \\
y' &= y - axy
\end{align*}
\]

there is a fixed point at \(x = 1, y = 0\). Show this, and then determine the stability of this fixed point. Your answer should depend on the value of \(a\).

To show that \(x = 1, y = 0\) is a fixed point, we need only substitute these values into the right-hand sides of the differential equations. We get \(-1(1-1) + 0 = 0\) and \(0 - a(1)(0) = 0\).

The Jacobian matrix for this system would be \(J(x, y) = \begin{bmatrix} -1 + 2x & 1 \\ -ay & 1 - ax \end{bmatrix}\). When the Jacobian is evaluated at \(x = 1, y = 0\) we get \(J(1, 0) = \begin{bmatrix} 1 & 1 \\ 0 & 1 - a \end{bmatrix}\) so that the trace and determinant at \(x = 1, y = 0\) are \(tr(J(1, 0)) = 2 - a\) and \(det(J(1, 0)) = 1 - a\). Since we know that a fixed point is a saddle if the determinant is negative, the fixed point at \((1, 0)\) is a saddle if \(1 - a < 0\), that is, if \(a > 1\). For \(a < 1\), we have \(det(J(1, 0)) > 0\) and \(tr(J(1, 0)) > 1\). Thus the point \((1, 0)\) is either a source or spiral source. To be a spiral source we would need \(det(J(1, 0)) > \frac{1}{4} tr(J(1, 0))^2\), that is \(1 - a > \frac{1}{4}(2 - a)^2\). Solving this inequality we get \(a^2 < 0\). Since this is not true for any real value of \(a\), the point \((1, 0)\) is never a spiral source. Hence \((1, 0)\) is a source for \(a < 1\). Note: at \(a = 0\) the \((\text{trace, det})\) point lies on the parabola \(det = \frac{1}{4}tr^2\), which in turn corresponds to equal (and in this case positive) eigenvalues, which still results in a (non-spiral) source. In Figure 5.4 we show phase portraits corresponding to values of the parameter \(a\) that are below, at, and above the value \(a = 1\).
In many cases we want to determine the fixed points and their stability as the parameter varies. As with first-order differential equations, for systems of first-order equations we say that a bifurcation occurs when the number of fixed points, or when the stability type of one or more fixed points, changes as the parameter goes through some value (called the bifurcation value). Thus in Example 5.4.1 we have a bifurcation at $a = 1$.

In terms of the trace-determinant plane, we are looking for values of the parameter where the point $(\text{trace}; \text{det})$ crosses from one of the key regions of the trace-determinant plane into another ($\text{trace}$ and $\text{det}$ refer to the trace and determinant of the Jacobian at a given fixed point). We illustrate these ideas with two more examples.

**Example 5.4.2** Consider the damped pendulum model from the exercises of the last section, given by

$$
\begin{align*}
x' &= y \\
y' &= -4\sin(x) - ay.
\end{align*}
$$

where we have added the parameter $a$ to represent the amount of damping in the system. For $a > 0$ there is positive damping, and for $a < 0$ there is negative damping. Negative damping is not physically realistic in a real pendulum (it tends to speed up the pendulum in the direction that it is already moving, rather than slowing it down), but we can still study the mathematics for $a$ either positive or negative.

Determine the bifurcation values in terms of the parameter $a$, that is, determine the values of $a$ for which the number of fixed points or the stability type of any of the fixed points changes.

First we must determine the equilibrium points in terms of the parameter $a$. We get $y = 0$
and \(-4\sin(x) - ay = 0\). This implies that \(\sin(x) = 0\), and so the equilibrium values are given by \((0, 0), (\pm \pi, 0), (\pm 2\pi, 0), \ldots\) or equivalently at \((\pm n\pi, 0)\) where \(n\) can be any nonnegative integer.

Next we find the Jacobian at each equilibrium point. We have 
\[
J = \begin{bmatrix}
0 & 1 \\
-4\cos(x) & -a
\end{bmatrix}.
\]
At the points \((\pm n\pi, 0)\) for \(n\) even we get 
\[
J_{n\text{ even}}(\pm n\pi, 0) = \begin{bmatrix}
0 & 1 \\
-4 & -a
\end{bmatrix}
\]
and for \(n\) odd we get 
\[
J_{n\text{ odd}}(\pm n\pi, 0) = \begin{bmatrix}
0 & 1 \\
4 & -a
\end{bmatrix}.
\]

Now find the trace and determinant for each case. For both cases we get 
\[\text{trace}(J) = -a.\]
For \(n\) even we get \(\det(J_{n\text{ even}}) = 4\) and for \(n\) odd we get \(\det(J_{n\text{ odd}}) = -4\). Thus, for \(n\) odd we always have a saddle point. For \(n\) even, we can have a source, spiral source, spiral sink or sink. Thus there will be 3 values of \(a\) at which bifurcations occur, all involving the fixed points at \((\pm n\pi, 0)\) for \(n\) even.

To find the bifurcation points, we only need to look at 
\[
J_{n\text{ even}}(\pm n\pi, 0) = \begin{bmatrix}
0 & 1 \\
-4 & -a
\end{bmatrix}.
\]
We have 
\[\text{trace} \begin{bmatrix}
0 & 1 \\
-4 & -a
\end{bmatrix} = -a\]
and 
\[\det \begin{bmatrix}
0 & 1 \\
-4 & -a
\end{bmatrix} = 4.\]
Letting \(a\) vary from \(-\infty\) to \(\infty\), the first bifurcation occurs when the point \((\text{trace}, \det) = (-a, 4)\) crosses from the source region to the spiral source region. This would be when \(4 = \frac{1}{4}a^2\) (with \(a < 0\)), and hence when \(a = -4\). The next bifurcation occurs when the point \((\text{trace}, \det) = (-a, 4)\) crosses from the spiral source region to the spiral sink region, which is when \(\text{trace} = -a = 0\), and hence \(a = 0\). The last bifurcation occurs when the point \((\text{trace}, \det) = (-a, 4)\) crosses from the spiral sink region to the sink region, which is when \(4 = \frac{1}{4}a^2\) again, but this time with \(a > 0\), and so \(a = 4\). We illustrate with a number line for the parameter \(a\) in Figure 5.5.

**Figure 5.5:** Parameter number line for bifurcations of a damped pendulum

It also helps to see what is going on here using a trace-determinant plot. In Figure 5.6 the points along the horizontal line at \(\det = 4\) represent the trace and determinant of 
\[
J_{n\text{ even}}(\pm n\pi, 0)
\]
for various values of \(a\). Note that this line is essentially the number line from Figure 5.5, after being “flipped” (because the trace of \(J_{n\text{ even}}(\pm n\pi, 0)\) is \(-a\)).
In Figure 5.7 we show phase portraits for values of $a$ at and between the bifurcation values $a = -4$, $a = 0$, and $a = 4$.

**Example 5.4.3** The system $x' = ax(1 - x) - xy$, $y' = -\frac{1}{2}y + xy$, could be used as a model for a predator-prey system of two species, with $x$ representing the prey, and $y$ the predators. For this system, in the absence of predators, we are left with the single equation $x' = ax(1 - x)$. For $a$ positive this represents logistic growth with carrying capacity of 1.
unit; for \( a \) negative it represents a model where for any initial \( x \) value for which \( 0 < x < 1 \), the solution goes asymptotically to zero, and for any initial \( x \) value for which \( x > 1 \), the solution increases to infinity. In the absence of prey, we get only the equation \( y' = -\frac{1}{2}y \) for which the solution goes to zero for any initial condition.

For this system, determine all fixed points, and then determine all bifurcations for the entire system (consider all fixed points) in terms of the parameter \( a \). Discuss the results in terms of the predator-prey system.

Solving \( ax(1 - x) - xy = 0 \) and \(-\frac{1}{2}y + xy = 0 \) simultaneously we get the fixed points \((0, 0)\), \((1, 0)\), and \((\frac{1}{2}, \frac{1}{2}a)\). The first fixed point represents no predators or prey, the second prey only, and for positive \( a \), the third fixed point represents predators and prey coexisting. When \( a \) is negative the third fixed point is not physically realistic. The general Jacobian matrix would be:

\[
J(x, y) = \begin{bmatrix} a(1 - 2x) - y & -x \\ y & -\frac{1}{2} + x \end{bmatrix}.
\]

We consider the fixed point \((0, 0)\) first. The Jacobian for this fixed point is \( J(0, 0) = \begin{bmatrix} a & 0 \\ 0 & -\frac{1}{2} \end{bmatrix} \). Since \( \det(J(0, 0)) = -\frac{1}{2}a < 0 \) for \( a > 0 \), the origin is a saddle for positive \( a \). For \( a < 0 \), \( \det(J(0, 0)) > 0 \) and \( \text{tr}(J(0, 0)) = a - \frac{1}{2} < 0 \) so the origin is a sink of some type. The inequality \( \det > \frac{1}{4}\text{tr}^2 \) (the condition for a spiral sink) for this fixed point becomes \(-\frac{1}{2}a > \frac{1}{4}(a - \frac{1}{2})^2 \) which is equivalent to \((a + \frac{1}{2})^2 < 0\), and hence has no solutions. Thus for negative \( a \) the origin is a sink, and the bifurcation point is at \( a = 0 \).

Next we look at the fixed point \((1, 0)\). The Jacobian is \( J(1, 0) = \begin{bmatrix} -a & -1 \\ 0 & \frac{1}{2} \end{bmatrix} \) so that \( \text{tr}(J(1, 0)) = \frac{1}{2} - a \) and \( \det(J(1, 0)) = -\frac{1}{2}a \). Thus we get a saddle for \( a > 0 \) and source of some type for \( a < 0 \) (\( \det > 0 \) and \( \text{tr} > 0 \)). The condition \( \det > \frac{1}{4}\text{tr}^2 \) for a spiral source becomes \(-\frac{1}{2}a > \frac{1}{4}(\frac{1}{2} - a)^2 \) which, as we saw above, has no solutions. Therefore we get a source for \( a < 0 \), and again, the bifurcation point is at \( a = 0 \).

Finally, for the fixed point \((\frac{1}{2}, \frac{1}{2}a)\) we have \( J(\frac{1}{2}, \frac{1}{2}a) = \begin{bmatrix} -\frac{1}{2}a & -\frac{1}{2} \\ \frac{1}{2}a & 0 \end{bmatrix} \), \( \text{tr}(J(\frac{1}{2}, \frac{1}{2}a)) = -\frac{1}{2}a \) and \( \det(J(\frac{1}{2}, \frac{1}{2}a)) = \frac{1}{4}a \). For \( a < 0 \) we get a negative determinant and hence a saddle. For \( a > 0 \), \( \det > 0 \) and \( \text{tr} < 0 \) and therefore some type of sink. The condition for a spiral sink is \( \frac{1}{4}a > \frac{1}{4}(-\frac{1}{2}a)^2 \), or equivalently, \( a(a - 4) < 0 \). This will be true if \( 0 < a < 4 \). Thus for \( 0 < a < 4 \) we get a spiral sink and for \( a > 4 \) we get a sink. The bifurcation points are \( a = 0 \) and \( a = 4 \).
To summarize the above results, there are 2 bifurcation points in terms of the parameter $a$. As $a$ increases we find that at $a = 0$ the fixed point at $(0, 0)$ changes from a sink to a saddle, the fixed point at $(1, 0)$ changes from a source to a saddle, and the fixed point $(\frac{1}{2}, \frac{1}{2}a)$ changes from a saddle to a spiral sink. At $a = 4$ the fixed point at $(\frac{1}{2}, \frac{1}{2}a)$ changes from a spiral sink to a sink. See Figure 5.8 for phase portraits representing values of $a$ between and on either side of the bifurcation points.

![Figure 5.8: Phase portraits for bifurcations of a predator-prey model](image)

The parameter $a$ represents the small-population growth (or decay) rate for the prey in the absence of predators. For positive values of $a$ all positive initial conditions will eventually go to the equilibrium point at $(\frac{1}{2}, \frac{1}{2}a)$. The larger the value of $a$, the higher the final population value of the predators will become, though the prey value will always settle at $\frac{1}{2}$. For positive values of $a$ less than $4$ the populations will oscillate as they approach the final values; for values of $a$ greater than $4$ the populations do not oscillate.

For negative values of $a$, the absence of predators (that is, $y = 0$) results in a prey population which goes to zero asymptotically if it starts out below $1$, and increases without bound if it starts above $1$. For this case, for all positive initial values both populations go extinct (asymptotically). Note that when there are few predators initially, and the initial prey population is above $1$, the prey increase dramatically at first, but eventually enough predators enter to drive the system back to extinction.

**Exercises** For each system below, find all equilibrium points, and bifurcation values for the given parameter

1. $x' = y$, $y' = -ax - y$
2. \( x' = y, \quad y' = -x - by \)

The differential equation \( x'' + ax' + x^3 - bx = 0 \) can be used to model a “double well potential” mechanical system (in the literature it is referred to as a Duffing equation). An example of such a system would be an upright flexible beam, fixed at the bottom but free at the top, with \( x \) measuring the distance from the vertical at the top of the beam. If the beam is flexible enough, it will flop to the left or right and come to rest there. The parameter \( a \) represents the amount of damping in the system, and \( b \) is a measure how far the beam will flop to either side when it comes to rest (a measure of the rigidity of the beam). When put into system form this equation becomes \( x' = y, \quad y' = -ay - x^3 + bx \) (show this). For the next two problems below, find all equilibrium points for each system in terms of the given parameter, and then find the trace and determinant of the Jacobian matrix for each fixed point. Using this information find all bifurcation values in terms of the given parameter for each system. Explain what this means in terms of the mechanical system. Consider all values of each parameter, even physically unrealistic ones.

3. \( x' = y, \quad y' = -ay - x^3 + x \)

4. \( x' = y, \quad y' = -y - x^3 + bx \)

The system \( x' = x(1 - x) - axy, \quad y' = -y + bxy \) can be used as a model for a predator-prey system, with \( x \) representing the number of prey, and \( y \) representing the number of predators (see the third example from this section). The parameter \( a \) represents the degree to which interactions between the species subtracts from the prey, and \( b \) represents the degree to which interactions between the species adds to the predators. For the next two problems below, find all equilibrium points for each system in terms of the given parameter, and then find the trace and determinant of the Jacobian matrix for each fixed point. Using this information find all bifurcation values in terms of the given parameter for each system. Explain what this means in terms of the predator-prey system. Consider all values of each parameter, even physically unrealistic ones.

5. \( x' = x(1 - x) - axy, \quad y' = -y + xy \)

6. \( x' = x(1 - x) - xy, \quad y' = -y + bxy \)
Chapter 6

Laplace Transforms

In this chapter we introduce yet another method for solving linear differential equations. The method applies to linear equations of any order. Although we already have methods for solving linear equations with constant coefficients, the method of Laplace Transforms gives the solution in a slightly different form: instead of arbitrary constants, the solution contains the initial conditions directly. This method is also very useful for non-homogeneous problems where the forcing function is defined piecewise, or contains an impulse function (impulse functions have not been encountered yet). Finally, this method is useful for interpreting solutions to mass-spring problems and to electrical circuit problems.

6.1 Simple Laplace Transforms

In this section we introduce the concept of the Laplace transform, and find a few useful, but elementary, transforms. We then demonstrate how to solve differential equations using Laplace transforms, in cases where the differential equation is relatively simple.

**Definition** 6.1.1 The Laplace transform of a function $f(t)$, $t > 0$, is a new function $F(s)$ defined as

$$F(s) = \int_{0}^{\infty} e^{-st} f(t) dt.$$

We also use the notation

$$F(s) = L[f(t)]$$

where $L$ is called the Laplace operator.
The Laplace transform turns a function of \( t \) into a different function of \( s \); or, as an engineer might say, it transforms a function from the \( t \)-domain into the \( s \)-domain. We will use the convention that lower case letters such as \( f \), \( g \), and \( h \), represent functions of \( t \), and the corresponding upper case letters \( F \), \( G \), and \( H \), represent the Laplace transforms of those functions (and hence are functions of \( s \)). Our goal is first to build a table of Laplace transforms for the functions that come up frequently in differential equations; exponential functions, trig functions, and polynomials. We will also need a few general properties of Laplace transforms, that apply to arbitrary functions. It will then be shown how to solve certain differential equations using these new tools.

Note that the integral, in the definition of the Laplace transform, is an improper integral; and in Calculus you were taught to evaluate it as a limit:

\[
\int_0^\infty e^{-st} f(t) \, dt = \lim_{B \to \infty} \int_0^B e^{-st} f(t) \, dt.
\]

When working with Laplace transforms, we will require that any function \( f(t) \) that is to be transformed must be at least piecewise continuous and of exponential order.

**Definition 6.1.2** A function \( f(t) \) is said to be of exponential order if there exist positive constants \( M, a \) and \( T \) such that

\[
|f(t)| < Me^{at} \quad \text{for all} \quad t \geq T.
\]

With this assumption, the integral \( \int_0^\infty e^{-st} f(t) \, dt \) and expressions such as \( e^{-st} f(t) \bigg|_{t=0}^{t=\infty} \) will always be defined for \( s \) large enough; that is, we will not have to worry about the existence of limits as \( t \to \infty \).

**Theorem 6.1** The Laplace transforms of the functions \( e^{at} \), \( \cos(bt) \), \( \sin(bt) \), and \( t^n \), as well as the constant function \( c \), are given in the table below. In the table \( a \), \( b \) and \( c \) can be any real numbers, and \( n \geq 0 \) is an integer.

<table>
<thead>
<tr>
<th>formula</th>
<th>function ( f(t) )</th>
<th>Laplace transform ( F(s) = L[f(t)] )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( c )</td>
<td>( \frac{c}{s} ), ( s &gt; 0 )</td>
</tr>
<tr>
<td>2</td>
<td>( e^{at} )</td>
<td>( \frac{1}{s-a} ), ( s &gt; a )</td>
</tr>
<tr>
<td>3</td>
<td>( \cos(bt) )</td>
<td>( \frac{s}{s^2 + b^2} ), ( s &gt; 0 )</td>
</tr>
<tr>
<td>4</td>
<td>( \sin(bt) )</td>
<td>( \frac{b}{s^2 + b^2} ), ( s &gt; 0 )</td>
</tr>
<tr>
<td>5</td>
<td>( t^n )</td>
<td>( \frac{n!}{s^{n+1}} ), ( s &gt; 0 )</td>
</tr>
</tbody>
</table>
Proof: The first two formulas are quite easy to establish, and we will prove them here. The proofs of the next three will be done in the exercises. For 1 we have

\[ F(s) = L[c] = \int_{0}^{\infty} e^{-st} \, dt = -\frac{c}{s} e^{-st} \bigg|_{t=0}^{t=\infty} = 0 + \frac{c}{s} \]

provided that \( s > 0 \). Similarly for 2

\[ F(s) = L[e^{at}] = \int_{0}^{\infty} e^{-st} e^{at} \, dt = \int_{0}^{\infty} e^{(-s+a)t} \, dt = \frac{1}{-s + a} e^{(-s+a)t} \bigg|_{t=0}^{t=\infty} \]

\[ = \frac{1}{-s + a} e^{-\infty} - \frac{1}{-s + a} e^{0} = \frac{1}{s - a} \]

as long as \( s > a \) (and hence \(-s + a\) is negative).

In order to effectively use Laplace transforms we also need a few rules that apply to functions in general, rather than to specific functions.

**Theorem 6.2** The Laplace transforms in the table below apply to arbitrary functions \( f(t) \) and \( g(t) \), whose Laplace transforms are \( F(s) \) and \( G(s) \) respectively, and real number constants \( a \) and \( b \).

<table>
<thead>
<tr>
<th>formula</th>
<th>function</th>
<th>Laplace transform</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>( f(t) )</td>
<td>( F(s) )</td>
</tr>
<tr>
<td>7</td>
<td>( af(t) + bg(t) )</td>
<td>( aF(s) + bG(s) )</td>
</tr>
<tr>
<td>8</td>
<td>( f'(t) )</td>
<td>( sF(s) - f(0) )</td>
</tr>
<tr>
<td>9</td>
<td>( f''(t) )</td>
<td>( s^2F(s) - sf(0) - f'(0) )</td>
</tr>
</tbody>
</table>

Proof: Six in the table is just the definition of the Laplace transform, included here for completeness. Seven follows easily from the definition of the Laplace transform and the properties of integration:

\[ L[af(t) + bg(t)] = \int_{0}^{\infty} e^{-st} (af(t) + bg(t)) \, dt = a \int_{0}^{\infty} e^{-st} f(t) \, dt + b \int_{0}^{\infty} e^{-st} g(t) \, dt \]

\[ = aL[f(t)] + bL[g(t)] = aF(s) + bG(s). \]
Eight in the table is derived using integration by parts:

\[
L[f'(t)] = \int_0^\infty e^{-st}f'(t)dt = e^{-st}f(t)|_{t=0}^{t=\infty} - \int_0^\infty -se^{-st}f(t)dt
\]

\[
= 0 - f(0) + s \int_0^\infty e^{-st}f(t)dt = -f(0) + sF(s).
\]

provided that \(\lim_{t \to \infty} e^{-st}f(t) = 0\) for \(s > a\), for some \(a\) (and this can be seen to be true for any \(f(t)\) of exponential order). Nine can be proven using eight; we leave the details to the exercises.

Comment: Seven is a very important property, called \underline{linearity} (it implies that the Laplace operator is a \underline{linear} operator). Another way to state property seven using the Laplace operator is

\[
L[af(t) + bg(t)] = aL[f(t)] + bL[g(t)].
\]

The basic idea with linear operators is that you can “distribute” them, and then pull out constants.

Example 6.1.1 Find the Laplace transform of \(2e^{-3t} + 4\sin(5t) - 6t^7\).

Using linearity we have

\[
L[2e^{-3t} + 4\sin(5t) - 6t^7] = 2L[e^{-3t}] + 4L[\sin(5t)] - 6L[t^7].
\]

Next we apply formulas 2 \((a = -3)\), 4 \((b = 5)\), and 5 \((n = 7)\):

\[
2L[e^{-3t}] + 4L[\sin(5t)] - 6L[t^7] = 2\frac{1}{s+3} + 4\frac{5}{s^2 + 25} - 6\frac{7!}{s^8}
\]

\[
= \frac{2}{s+3} + \frac{20}{s^2 + 25} - \frac{30240}{s^8}
\]

Example 6.1.2 Find the Laplace transform of both sides of the differential equation \(y' - 2y = 3\cos(4t)\). Then equate the transforms of both sides and solve for \(Y(s)\), the transform of the solution \(y(t)\).

On the left-hand side we use linearity followed by formulas 6 and 8:

\[
L[y' - 2y] = L[y'] - 2L[y] = sY(s) - y(0) - 2Y(s).
\]
On the right-hand side, we use linearity followed by formula 3 \((b = 4)\):

\[
L[3 \cos(4t)] = 3L[\cos(4t)] = 3 \frac{s}{s^2 + 16}.
\]

Equating the transforms of both sides we get

\[
sY(s) - y(0) - 2Y(s) = \frac{3s}{s^2 + 16}.
\]

Then we move terms that do not have \(Y(s)\) in them to the right side, and factor out \(Y(s)\) on the left:

\[
Y(s)(s - 2) = \frac{3s}{s^2 + 16} + y(0).
\]

Finally, we divide by \((s - 2)\) and distribute to get

\[
Y(s) = \frac{3s + y(0)}{s^2 + 16} = \frac{3s}{(s^2 + 16)(s - 2)} + \frac{y(0)}{s - 2}.
\]

**Solving differential equations and the Inverse Laplace transform**

In example 6.1.2 we came close to solving the given differential equation; all we need to do is recover the solution \(y(t)\) from its transform \(Y(s)\). In order to go from the Laplace transform of a function back to the original function we must apply the inverse Laplace transform. We use the symbol \(L^{-1}\) to denote the inverse Laplace transform operator. Thus, if \(F(s) = L[f(t)]\), then \(f(t) = L^{-1}[F(s)]\). Just as the Laplace transform takes us from the left column to the right column for formulas 1-9 in Theorems 6.1 and 6.2, the inverse Laplace transform takes us from the right column to the left column.

**Theorem** 6.3 The inverse Laplace transform is a linear operator; that is, \(L^{-1}[aF(s) + bG(s)] = aL^{-1}[F(s)] + bL^{-1}[G(s)]\).

Proof. Apply \(L^{-1}\) to both sides of \(L[af(t) + bg(t)] = aL[f(t)] + bL[g(t)]\).

In order to apply the inverse Laplace transform effectively in solving differential equations, we need one more tool. Looking back at example 6.1.2, in order to finish off the problem and find the inverse Laplace transform of \(Y(s)\) we need to find \(L^{-1}\left[\frac{3s}{(s^2 + 16)(s - 2)}\right]\). Since there is no form in the right column of formulas 1-9 in Theorems 6.1 and 6.2 that corresponds
to this function of $s$, we need to apply a procedure that you may have encountered in a calculus course: \emph{partial fractions}. 
Review of partial fractions

The idea of partial fractions is to rewrite a rational function (a ratio of two polynomials), whose denominator is of higher degree than the numerator, by factoring the denominator, and expressing that rational function as a sum of new rational functions, whose denominators are the factors of the denominator of the original function. For example, \( \frac{1}{(x-1)(x-2)} \) can be rewritten as \( \frac{1}{x-2} - \frac{1}{x-1} \) (show this). In example 6.1.2, \( \frac{3s}{(s^2+16)(s^2)} \) can be rewritten as
\[
\frac{3}{s} + \frac{\left(\frac{12}{s^2+16}\right)}{s^2+16} = \frac{3}{s} + \frac{12}{5} \frac{1}{s^2+16} - \frac{3}{10} \frac{s}{s^2+16}
\]
after which the inverse Laplace transform can easily be found (see the next example).

Partial fractions procedure:

Step 1: Starting with a rational function \( P(s)/Q(s) \), with the degree of the polynomial \( Q(s) \) greater than that of \( P(s) \), factor the denominator into linear and non-factorable quadratic factors (we consider only real-valued factors). The single linear factors will have the form \( as + b \) and the quadratic factors will have the form \( as^2 + bs + c \), with \( b^2 - 4ac < 0 \) (if \( b^2 - 4ac \geq 0 \) then the quadratic factor can itself be factored into linear factors). Repeated linear factors will be of the form \( (as + b)^n \) for some integer \( n \geq 2 \).

Step 2: For the partial fraction expansion, assume the form \( \frac{A}{as+b} \) for each single linear factor in the denominator, assume the form \( \frac{Bs+C}{as^2+bs+c} \) for each single quadratic factor, and assume the form \( \frac{D}{as+b} + \frac{E}{(as+b)^2} + \ldots + \frac{Z}{(as+b)^n} \) for each repeated linear factor \( (as+b)^n \). We do not consider repeated quadratic factors, or any higher order factors, although these can be easily handled by computer algebra systems.

Step 3: Set the rational function from Step 1 equal to the sum of the assumed forms from Step 2, find a common denominator, and then equate and multiply out the numerators.

Step 4: Equate coefficients of like terms from the equation in Step 3. This will give you \( n \) linear equations in \( n \) unknowns \( A, B, C, \ldots \). Solve the set of equations in Step 3 using a standard technique such as Gaussian elimination, or even better, using a graphing calculator or computer program.

Example 6.1.3 Find the partial fraction expansion for \( \frac{s+2}{(s+1)(s-1)^2(s^2+1)} \).

Step 1 is done since the denominator is already factored as much as possible, so we go to Steps 2 and 3 and assume
\[
\frac{s+2}{(s+1)(s-1)^2(s^2+1)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+1} + \frac{D}{s-1} + \frac{E}{(s-1)^2}.
\]
The common denominator is \((s + 1)(s - 1)^2(s^2 + 1)\) so

\[
\frac{s + 2}{(s + 1)(s - 1)^2(s^2 + 1)} = \frac{A(s - 1)^2(s^2 + 1)}{(s + 1)(s - 1)^2(s^2 + 1)} + \frac{(Bs + C)(s + 1)(s - 1)^2}{(s + 1)(s - 1)^2(s^2 + 1)} + \frac{D(s + 1)(s - 1)(s^2 + 1)}{(s + 1)(s - 1)^2(s^2 + 1)} + \frac{E(s - 1)(s^2 + 1)}{(s + 1)(s - 1)^2(s^2 + 1)}
\]

or after equating and multiplying out the numerators

\[
s + 2 = A - 2As + 2As^2 - 2As^3 + As^4 + C + Bs - Cs - Bs^2 - Cs^2 - Bs^3 + Cs^3 + Bs^4 - D + s^4D + E + sE + s^2E + s^3E
\]

Finally we go to Step 4 and equate the coefficients of the various powers of \(s\) on both sides of the equation (as well as the constant terms without \(s\), as they can be considered coefficients of \(s^0 = 1\)). This results in the following 5 linear equations in 5 unknowns:

\[
\begin{align*}
s^4 & : 0 = A + B + D \\
s^3 & : 0 = -2A - B + C + E \\
s^2 & : 0 = 2A - B - C + E \\
s & : 1 = -2A + B - C + E \\
\text{constants (}s^0\text{)} & : 2 = A + C - D + E
\end{align*}
\]

Solving this system using the solve command of a graphing calculator or computer software we get

\[
A = \frac{1}{8}, B = \frac{3}{4}, C = \frac{1}{4}, D = -\frac{7}{8}, E = \frac{3}{4}
\]

thus we can write

\[
\frac{s + 2}{(s + 1)(s - 1)^2(s^2 + 1)} = \frac{1}{8} \cdot \frac{s}{s + 1} + \frac{3}{4} \cdot \frac{s + 1}{s^2 + 1} - \frac{7}{8} \cdot \frac{s}{s - 1} + \frac{3}{4} \cdot \frac{s}{(s - 1)^2}.
\]

The good news is that if we have a computer algebra system available, we can get partial fraction expansions directly. Most computer algebra systems have a command (such as expand, on the TI-89 or TI-92) which will take a rational function as argument and produce its partial fraction expansion. From this point forward in the text, we will not explicitly
derive partial fraction expansions, but will assume that the student is either comfortable
doing them by hand as in the previous example, or has a computer algebra system available
(preferably both).

**Three step procedure for solving linear differential equations using Laplace transforms:**

<table>
<thead>
<tr>
<th>Step 1</th>
<th>Take the Laplace transform of both sides of the differential equation.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Step 2</td>
<td>Solve for the transform of the solution ( Y(s) ).</td>
</tr>
<tr>
<td>Step 3</td>
<td>Find the inverse Laplace transform of ( Y(s) ) to get the solution ( y(t) ).</td>
</tr>
</tbody>
</table>

**Example 6.1.4** Use Laplace transforms to find the solution to the differential equation
\( y' - 2y = 3\cos(4t) \). This is the same differential equation as in Example 6.1.2.

In Example 6.1.2 we took the Laplace transform of both sides of the differential equation, and solved for \( Y(s) \) (Steps 1 and 2 above) to get

\[
Y(s) = \frac{3s}{s^2 + 16} + \frac{y(0)}{s - 2} = \frac{3s}{(s^2 + 16)(s - 2)} + \frac{y(0)}{s - 2}.
\]

In order to find \( L^{-1}[Y(s)] \) we need to do a partial fractions expansion. We know that the form of the expansion will be

\[
\frac{3s}{(s^2 + 16)(s - 2)} = \frac{A}{s - 2} + \frac{Bs + C}{s^2 + 16}.
\]

We can find the values of \( A \), \( B \), and \( C \) as in Example 6.1.3 (find a common denominator, equate numerators, multiply out, equate coefficients of like terms, solve the resulting linear system), or we can use technology (e.g. the expand command of the TI-89). In either case we get

\[
\frac{3s}{(s^2 + 16)(s - 2)} = \frac{3}{10} \frac{1}{s - 2} + \frac{-3/8 + 12/5}{s^2 + 16}.
\]

We next multiply out the second term, apply \( L^{-1} \), and use linearity to get

\[
L^{-1} \left[ \frac{3s}{(s^2 + 16)(s - 2)} \right] = \frac{3}{10} L^{-1} \left[ \frac{1}{s - 2} \right] - \frac{3}{10} L^{-1} \left[ \frac{s}{s^2 + 16} \right] + \frac{12}{5} \frac{1}{s^2 + 16}.
\]
The goal is now to get each expression in brackets above into a form where each term appears in the right column of the table of Laplace transforms in Theorem 6.1 (formulas 1-5). The first two terms correspond to formulas 2 and 3 respectively. The third term needs a minor adjustment to fit formula 4; we need $\sqrt{16} = 4$ in the numerator, so using linearity we write $L^{-1}\left[\frac{1}{s^2+16}\right] = \frac{1}{4}L^{-1}\left[\frac{4}{s^2+16}\right]$ to get

$$L^{-1}\left[\frac{3s}{(s^2+16)(s-2)}\right] = \frac{3}{10}L^{-1}\left[\frac{1}{s-2}\right] - \frac{3}{10}L^{-1}\left[\frac{s}{s^2+16}\right] + \frac{12}{5}L^{-1}\left[\frac{4}{s^2+16}\right]$$

$$= \frac{3}{10}e^{2t} - \frac{3}{10} \cos(4t) + \frac{3}{5} \sin(4t).$$

Finally, we get

$$y(t) = L^{-1}[Y(s)] = L^{-1}\left[\frac{3s}{(s^2+16)(s-2)} + \frac{y(0)}{s-2}\right]$$

$$= L^{-1}\left[\frac{3s}{(s^2+16)(s-2)}\right] + L^{-1}\left[\frac{y(0)}{s-2}\right]$$

$$= L^{-1}\left[\frac{3s}{(s^2+16)(s-2)}\right] + y(0)L^{-1}\left[\frac{1}{s-2}\right]$$

$$= \frac{3}{10}e^{2t} - \frac{3}{10} \cos(4t) + \frac{3}{5} \sin(4t) + y(0)e^{2t}.$$

**Example 6.1.5** Solve the initial value problem $y'' + y = \sin(2t)$, $y(0) = 3$, $y'(0) = 0$ using Laplace transforms.

Here we put it all together, and see the solution to a differential equation from beginning to end. Taking the Laplace transform of both sides and using linearity (formula 7, Theorem 6.2) we have

$$L[y''] + L[y] = L[\sin(2t)].$$

On the left side we use formulas 6 and 9 from Theorem 6.2, and on the right we use formula 4 from 6.1 to get

$$s^2Y(s) - sy(0) - y'(0) + Y(s) = \frac{2}{s^2 + 4}.$$
Putting in the initial conditions, and solving for $Y(s)$ we get

$$s^2Y(s) - 3s + Y(s) = \frac{2}{s^2 + 4}$$

$$Y(s)(s^2 + 1) = \frac{2}{s^2 + 4} + 3s$$

$$Y(s) = \frac{\frac{2}{s^2 + 4} + 3s}{s^2 + 1} = \frac{2}{(s^2 + 4)(s^2 + 1)} + \frac{3s}{s^2 + 1}$$

Using partial fractions on the first term, we assume

$$\frac{2}{(s^2+4)(s^2+1)} = \frac{As+B}{s^2+4} + \frac{Cs+D}{s^2+1}$$

and then solve for the constants, or use the expand command of a computer algebra system to get

$$\frac{2}{(s^2+4)(s^2+1)} = -\frac{\frac{2}{3}}{s^2+4} + \frac{\frac{2}{3}}{s^2+1}.$$ 

We now use the inverse Laplace transform to get

$$y(t) = L^{-1}[Y(s)] = L^{-1}\left[-\frac{\frac{2}{3}}{s^2+4} + \frac{\frac{2}{3}}{s^2+1} + \frac{3s}{s^2+1}\right]$$

$$= L^{-1}\left[-\frac{\frac{2}{3}}{s^2+4}\right] + L^{-1}\left[\frac{\frac{2}{3}}{s^2+1}\right] + L^{-1}\left[\frac{3s}{s^2+1}\right]$$

$$= -\frac{1}{3}L^{-1}\left[\frac{2}{s^2+4}\right] + \frac{2}{3}L^{-1}\left[\frac{1}{s^2+1}\right] + 3L^{-1}\left[\frac{s}{s^2+1}\right]$$

$$= -\frac{1}{3} \sin(2t) + \frac{2}{3} \sin(t) + 3 \cos(t).$$

You should determine which formulas were used at each step.

**Exercises** Find the Laplace transforms for the functions in 1-4 below, using formulas 1-5 from Theorem 6.1 and linearity.

1. $\cos(t) + 2e^t$.
2. $3 - 2t + 4t^2$.
3. $-\sin(3t) - 2t^5 + 5$
4. $3e^{-7t} + 4 \cos(8t) - 3 + 5t^2$
For exercises 5-8, find the Laplace transform of both sides of the differential equation, and then solve for $Y(s)$.

5. $y' = y + 2$

6. $y' + 2y = 4\sin(3t)$

7. $y'' + 4y = \cos(t)$

8. $y'' + 4y = \cos(2t)$

For exercises 9-12, find the inverse Laplace transform of $Y(s)$ to get $y(t)$.

9. $Y(s) = \frac{2}{s-5} - \frac{3s}{s^2+4} + \frac{1}{s^2}$

10. $Y(s) = \frac{2}{s+5} - \frac{3}{s^2+9} - \frac{10}{s^2}$

11. $Y(s) = \frac{2}{2s+5} - \frac{3s}{s^2+2} + \frac{1}{5s^2}$

12. $Y(s) = \frac{2}{3s-5} - \frac{3}{2s^2+4} + \frac{1}{3s}$

For exercises 13-16 solve the differential equation or initial value problem using the three step procedure given in the text.

13. $y'' + 4y = \cos(t), \quad y(0) = 2, \quad y'(0) = 0$

14. $y' + 2y = 4\sin(3t), \quad y(0) = 3$

15. $y'' + 4y = 0$

16. $y' = y + 2$

Use integration to prove the following formulas.

17. $L[\cos(bt)] = \frac{s}{s^2+b^2}$. (formula 3)

Note: If you are familiar with complex numbers, it is easier to find $L[e^{ibt}]$ and take its real part.

18. $L[\sin(bt)] = \frac{b}{s^2+b^2}$. (formula 4)

See note in the previous example.

19. $L[t^n] = \frac{n!}{s^{n+1}},$ for any integer $n \geq 0$. (formula 5)

Hint: first show that $L[t^0] \equiv L[1] = 1/s$ and then use integration by parts to show that $L[t^n] = \frac{n}{s}L[t^{n-1}]$. 

20. Apply formula 8 to the function \( f'(t) \), to prove the formula \( L[f''(t)] = s^2 F(s) - sf(0) - f'(0) \).
6.2 Deriving More Laplace Transforms

In order to solve more differential equations of interest (in the last section all of our second order equations were missing the \(y'\) term), we need a few more Laplace transform rules, and an algebraic technique that you likely first (and probably last) saw in high school.

We start with two rules that are of the general type; that is, they apply to general functions \(f(t)\) as do the rules from Theorem 6.2 in the previous section. We will then show how to derive new specific Laplace transform rules using these general rules.

**Theorem 6.4** The Laplace transforms in the table below apply to a general function \(f(t)\), whose Laplace transform is \(F(s)\), and to the arbitrary constant \(a\).

<table>
<thead>
<tr>
<th>formula</th>
<th>function (e^{at}f(t))</th>
<th>Laplace transform (F(s - a))</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>(t^n f(t))</td>
<td>((-1)^n F^{(n)}(s))</td>
</tr>
</tbody>
</table>

Note: \(F^{(n)}(s)\) represents the \(n\)th derivative of \(F(s)\) with respect to \(s\).

Proof: The proofs of formulas 10 and 11 are left to the exercises.

**Example 6.2.1** Derive rules for the Laplace transforms of \(e^{at} \cos(bt)\) and \(e^{at} \sin(bt)\).

We want to apply formula 10 with \(f(t) = \cos(bt)\), but we first need to find \(F(s)\). We get

\[
F(s) = L[f(t)] = L[\cos(bt)] = \frac{s}{s^2 + b^2}
\]

from formula 3 of Theorem 6.1. Now applying formula 10 we get

\[
L[e^{at} \cos(bt)] = F(s - a) = \frac{s - a}{(s - a)^2 + b^2}.
\]

Similarly

\[
L[e^{at} \sin(bt)] = \frac{b}{(s - a)^2 + b^2}
\]

where this time \(F(s) = \frac{b}{s^2 + b^2}\) from formula 4 of Theorem 6.1.
Example 6.2.2 Derive rules for the Laplace transforms of \( t \cos(bt) \) and \( t \sin(bt) \).

To find \( L[t \cos(bt)] \) we want to apply formula 11 with \( f(t) = \cos(bt) \); we have \( F(s) = L[f(t)] = L[\cos(bt)] = \frac{s}{s^2 + b^2} \) as in the previous example. Now applying formula 11 with \( n = 1 \) we get

\[
L[t \cos(bt)] = \frac{-1}{s^2 + b^2}
\]

Similarly to find \( L[t \sin(bt)] \) we use \( F(s) = \frac{b}{s^2 + b^2} \) and hence

\[
L[t \sin(bt)] = \frac{-b}{s^2 + b^2}
\]

Example 6.2.3 Derive a rule for the Laplace transform of \( t^ne^{at} \).

To find \( L[t^ne^{at}] \) we choose to apply formula 10 from above with \( f(t) = t^n \). We have \( F(s) = L[f(t)] = L[t^n] = \frac{n!}{s^{n+1}} \) from Theorem 6.1, formula 5, and so applying formula 10 we get

\[
L[t^ne^{at}] = F(s - a) = \frac{n!}{(s - a)^{n+1}}.
\]

Note: We could alternatively have used formula 11 from above; we leave this approach to the exercises.

We summarize the results of the last three examples in Table form:
Example 6.2.4 Solve the initial value problem \( y'' + 2y' + y = 0, \quad y(0) = 3, \quad y'(0) = 0 \) using Laplace transforms.

We use the three step process: take the Laplace transform of both sides, solve for \( Y(s) \), then find the inverse Laplace transform of \( Y(s) \) to get \( y(t) \). Applying the Laplace operator \( L \) to both sides of the differential equation, using linearity, and substituting 3 for \( y(0) \) and 0 for \( y'(0) \) we get

\[
L[y'' + 2y' + y] = L[y''] + 2L[y'] + L[y] = s^2Y(s) - sy(0) - y'(0) + 2(sY(s) - y(0)) + Y(s) = s^2Y(s) - 3s + 2sY(s) - 6 + Y(s) = L[0] = 0.
\]

Then solving the last line above for \( Y(s) \) we get

\[
s^2Y(s) + 2sY(s) + Y(s) = 3s + 6
\]

thus \( Y(s) \left( s^2 + 2s + 1 \right) = 3s + 6 \) and so \( Y(s) = \frac{3s + 6}{s^2 + 2s + 1} \)

We now factor the denominator \( s^2 + 2s + 1 = (s + 1)^2 \) and expand the fraction into two terms to get

\[
Y(s) = \frac{3s + 6}{(s + 1)^2} = \frac{3s}{(s + 1)^2} + \frac{6}{(s + 1)^2}.
\]

Neither of the two terms of \( Y(s) \) corresponds to the right column of our Laplace transform tables exactly as is, so we must make some adjustments. For the first term \( \frac{3s}{(s+1)^2} \) we observe that if instead of \( s \) in the numerator we had \( (s + 1) \), then we could cancel and get a match with formula 2 from the previous section. Thus we both add and subtract the number 1 as
follows:

\[
\frac{3s}{(s+1)^2} = \frac{3(s+1-1)}{(s+1)^2} = \frac{3(s+1)-3}{(s+1)^2} \\
= \frac{3(s+1)}{(s+1)^2} - \frac{3}{(s+1)^2} \\
= \frac{3}{(s+1)} - \frac{3}{(s+1)^2}
\]

We now can combine some terms to get

\[
Y(s) = \frac{3s}{(s+1)^2} + \frac{6}{(s+1)^2} \\
= \frac{3}{(s+1)} - \frac{3}{(s+1)^2} + \frac{6}{(s+1)^2} \\
= \frac{3}{(s+1)} + \frac{3}{(s+1)^2}
\]

Finally we use linearity to get

\[
y(t) = L^{-1}[Y(s)] = L^{-1}\left[\frac{3}{(s+1)} + \frac{3}{(s+1)^2}\right] \\
= 3L^{-1}\left[\frac{1}{(s+1)}\right] + 3L^{-1}\left[\frac{1}{(s+1)^2}\right] \\
= 3e^{-t} + 3te^{-t}
\]

using formula 2 with \(a = -1\) on the first term and formula 16 with \(n = 1\) and \(a = -1\) on the second.

Before going much further, we need to review a technique that you probably first encountered in high school, called “completing the square”.

**Review of Completing the Square**

The idea of completing the square is to write a quadratic factor as the sum of a squared linear factor and a constant (both possibly multiplied by another constant). We illustrate how to complete the square using \(2s^2 + 16s + 64\) as an example:

1. Factor out the lead coefficient \(a = 2\) to get \(2(s^2 + 8s + 32)\).
2. Take half of the coefficient of \( s \) in step 1 (\( \frac{1}{2} \times 8 = 4 \) in this case) and then square that number (\( 4^2 = 16 \)).

3. Add and subtract the result from step 2 inside the expression in step 1 to get \( 2(s^2 + 8s + 16 - 16 + 32) \).

4. The first three terms inside the parentheses from step 3 factor as a perfect square \( s^2 + 8s + 16 = (s + 4)^2 \), and the last two terms combine as \(-16 + 32 = 16\), so that we now have \( 2((s + 4)^2 + 16) \).

This technique is used when you have a quadratic expression that cannot be factored, appearing in the denominator of an expression for which you want to find the inverse Laplace transform. For example, suppose that you want to find \( L^{-1} \left[ \frac{1}{s^2 + 2s + 2} \right] \). Since \( \frac{1}{s^2 + 2s + 2} \) does not directly correspond to the right column of our Laplace formulas, and since \( s^2 + 2s + 2 \) cannot be factored (using real numbers), we need to rewrite the expression as \( \frac{1}{s^2 + 2s + 2} = \frac{1}{s^2 + 2s + 1 + 1} = \frac{1}{(s+1)^2 + 1} \). Then \( L^{-1} \left[ \frac{1}{(s+1)^2 + 1} \right] = e^{-t} \sin t \) from formula 13 with \( a = -1 \) and \( b = 1 \).

**Note:** An irreducible quadratic expression (i.e., one that cannot be factored into the product of two linear factors) is an expression of the form \( as^2 + bs + c \) for which \( b^2 - 4ac \) is negative. You should factor quadratics that can be factored, and complete the square on irreducible quadratic expressions.

**Example** 6.2.5 Solve the differential equation \( y'' + 4y' + 8y = 0 \) using Laplace Transforms. Compare the form of the solution obtained to the form that would be obtained using the techniques of Chapter 3.

Taking the Laplace transform of both sides we get

\[
(s^2Y - sy(0) - y'(0)) + 4(sY - y(0)) + 8Y = 0.
\]
Then we solve for $Y$ as follows:

\[ s^2Y + 4sY + 8Y = sy(0) + y'(0) + 4y(0) \]

thus \((s^2 + 4s + 8)Y = sy(0) + y'(0) + 4y(0)\)

and so

\[
Y = \frac{sy(0)}{(s^2 + 4s + 8)} + \frac{y'(0)}{(s^2 + 4s + 8)} + \frac{4y(0)}{(s^2 + 4s + 8)}.
\]

The denominator of each expression $s^2 + 4s + 8$ must either be factored and then expanded (partial fractions) or it must be transformed by completing the square. Since $b^2 - 4ac = 4^2 - (4)(8) = -16$ we know that it cannot be factored, so we complete the square. Since

\[
s^2 + 4s + 8 = s^2 + 4s + 4 - 4 + 8 = (s + 2)^2 + 4
\]

we get

\[
Y = \frac{sy(0)}{(s + 2)^2 + 4} + \frac{y'(0)}{(s + 2)^2 + 4} + \frac{4y(0)}{(s + 2)^2 + 4}.
\]

Then we take the inverse Laplace transform of both sides, and using linearity we get

\[
y(t) = L^{-1}[Y] = y(0)L^{-1}\left[\frac{s}{(s + 2)^2 + 4}\right] + y'(0)L^{-1}\left[\frac{1}{(s + 2)^2 + 4}\right] + 4y(0)L^{-1}\left[\frac{1}{(s + 2)^2 + 4}\right].
\]

The first term in Equation 6.1 can be transformed to fit formula 12 (add and subtract $2$), and the second and third can be transformed to fit formula 13 (multiply and divide by $2$). For the first term

\[
y(0)L^{-1}\left[\frac{s}{(s + 2)^2 + 4}\right] = y(0)L^{-1}\left[\frac{s + 2 - 2}{(s + 2)^2 + 4}\right]
\]

\[
= y(0)L^{-1}\left[\frac{s + 2}{(s + 2)^2 + 4}\right] - y(0)L^{-1}\left[\frac{2}{(s + 2)^2 + 4}\right]
\]

\[
= y(0)e^{-2t}\cos 2t - y(0)e^{-2t}\sin 2t
\]

using formulas 13 and 12 respectively (with $a = -2$ and $b = 2$). For the second term in
Equation 6.1 we again use formula 12 (with \(a = -2\) and \(b = 2\)) to get

\[
y'(0)L^{-1}\left[\frac{1}{(s + 2)^2 + 4}\right] + 4y(0)L^{-1}\left[\frac{1}{(s + 2)^2 + 4}\right]
= y'(0)\left(\frac{1}{2}\right)L^{-1}\left[\frac{2}{(s + 2)^2 + 4}\right]
+ 4\left(\frac{1}{2}\right) y(0)L^{-1}\left[\frac{2}{(s + 2)^2 + 4}\right]
= \frac{1}{2}y'(0)e^{-2t}\sin(2t) + 2y(0)e^{-2t}\sin(2t).
\]

Putting all three terms of Equation 6.1 together we get

\[
y(t) = L^{-1}[Y] = y(0)e^{-2t}\cos(2t) - y(0)e^{-2t}\sin(2t)
+ \frac{1}{2}y'(0)e^{-2t}\sin(2t) + 2y(0)e^{-2t}\sin(2t).
\]

In Chapter 3, to solve \(y'' + 4y' + 8y = 0\), we would have used the roots of the characteristic polynomial \(r^2 + 4r + 8\) to write down the general solution. (Do you see a relationship between this characteristic polynomial and the polynomial \(Q(s) = s^2 + 4s + 8\) in the denominator of \(Y(s)\) in the Laplace transform method?) In this case the roots are \(-2 - 2i\) and \(-2 + 2i\), so the general solution would be \(C_1e^{-2t}\cos 2t + C_2e^{-2t}\sin 2t\). By factoring the Laplace solution we could write

\[
y(t) = y(0)e^{-2t}\cos 2t + 
\left(-y(0) + \frac{1}{2}y'(0) + 2y(0) = y(0) + \frac{1}{2}y'(0)\right) e^{-2t}\sin 2t
\]

which shows that \(C_1 = y(0)\) and \(C_2 = -y(0) + \frac{1}{2}y'(0) + 2y(0) = y(0) + \frac{1}{2}y'(0)\). The solution with the arbitrary constants \(C_1\) and \(C_2\) shows clearly the important concept that with a second order linear differential equation you need to find two linearly independent solutions. The Laplace solution also has two constants, \(y(0)\) and \(y'(0)\), and the Laplace form shows how the solution depends on the initial conditions.

We conclude this section with an example that requires several of the techniques that we have encountered in the last two sections. The reader is invited to determine the details of which Laplace formulas and techniques are used at each step.

**Example 6.2.6** Solve the initial value problem \(y'' + 4y' + 8y = 2\cos(t),\ y(0) = 0,\ y'(0) = 1\), using Laplace Transforms.
Take the Laplace transform of both sides to get

\[(s^2Y - sy(0) - y'(0)) + 4(sY - y(0)) + 8Y = \frac{2s}{s^2 + 1}\]

so that

\[(s^2 + 4s + 8)Y = \frac{2s}{s^2 + 1} + 1\]

and hence

\[Y = \frac{2s}{(s^2 + 1)(s^2 + 4s + 8)} + \frac{1}{s^2 + 4s + 8}\]

Now apply partial fractions to the first term (expand it) to get

\[Y = \frac{\frac{14}{65}s + \frac{8}{65}}{s^2 + 1} + \frac{-\frac{14}{65}s - \frac{64}{65}}{(s + 2)^2 + 4} + \frac{1}{(s + 2)^2 + 4}\]

Since \(s^2 + 4s + 8\) is irreducible we complete the square to get

\[Y = \frac{\frac{14}{65}s + \frac{8}{65}}{s^2 + 1} + \frac{-\frac{14}{65}s - \frac{64}{65}}{(s + 2)^2 + 4} + \frac{1}{(s + 2)^2 + 4} \quad (6.2)\]

We can handle the first term in Equation 6.2 by writing it as a sum of two terms \(\frac{\frac{14}{65}s + \frac{8}{65}}{s^2 + 1} = \frac{14}{65} \frac{s}{s^2 + 1} + \frac{8}{65} \frac{1}{s^2 + 1}\) and using formulas 3 and 4. The second term in Equation 6.2 is handled by expanding it and then applying the “add and subtract” trick to get it into a form where we can apply formulas 12 and 13.

\[
\begin{align*}
\frac{-\frac{14}{65}s - \frac{64}{65}}{(s + 2)^2 + 4} &= \frac{-\frac{14}{65}s}{(s + 2)^2 + 4} - \frac{\frac{64}{65}}{(s + 2)^2 + 4} \\
&= \frac{-\frac{14}{65}(s + 2)}{(s + 2)^2 + 4} + \frac{\frac{64}{65}}{(s + 2)^2 + 4} \\
&= \frac{-\frac{14}{65} - \frac{64}{65}}{1} \\
&= \frac{-\frac{14}{65}(s + 2)}{(s + 2)^2 + 4} + \frac{\frac{64}{65}}{(s + 2)^2 + 4} \\
&= \frac{-\frac{14}{65} - \frac{64}{65}}{1} \\
&= \frac{-\frac{14}{65}(s + 2) + \frac{64}{65}}{(s + 2)^2 + 4} \\
&= \frac{\frac{36}{65}}{(s + 2)^2 + 4}
\end{align*}
\]
Equation 6.2 now becomes

$$Y = \frac{14}{65} \frac{s}{s^2 + 1} + \frac{8}{65} \frac{1}{s^2 + 1} - \frac{14}{65} \frac{s + 2}{(s + 2)^2 + 4} - \frac{36}{65} \frac{1}{(s + 2)^2 + 4} + \frac{1}{(s + 2)^2 + 4}$$ (6.3)

Finally we combine the last two terms of Equation 6.3 and then apply the “multiply and divide” trick to get

$$Y = \frac{14}{65} \frac{s}{s^2 + 1} + \frac{8}{65} \frac{1}{s^2 + 1} - \frac{14}{65} \frac{s + 2}{(s + 2)^2 + 4} + \frac{29}{65} \frac{1}{(s + 2)^2 + 4}$$

Taking the inverse Laplace transform, and using the appropriate Laplace formulas (3, 4, 12, 13) we get

$$y(t) = \frac{14}{65} \cos(t) + \frac{8}{65} \sin(t) - \frac{14}{65} e^{-2t} \cos(2t) + \frac{29}{130} e^{-2t} \sin(2t).$$

**Exercises** Solve each initial value problem 1-4 using Laplace transforms.

1. \(y'' + 2y' + 2y = 0, \ y(0) = 1, \ y'(0) = 0\)
2. \(y'' - 4y' + 8y = 0, \ y(0) = 1, \ y'(0) = 0\)
3. \(y'' + 4y' + 8y = \cos(4t), \ y(0) = 0, \ y'(0) = 0\)
4. \(y'' + 4y' + 4y = 4e^{-2t}, \ y(0) = 1, \ y'(0) = 0\)

Use Laplace transforms to solve each of the differential equations 5 - 8, in terms of the initial conditions \(y(0)\) and \(y'(0)\). Compare your solution to the general solution you would obtain using the methods of Chapter 3 (identify the constants \(C_1\) and \(C_2\)).

5. \(x'' + 6x' + 9x = 0\)
6. \(y'' + 2y' + 5y = 0\)
7. \(y'' + y = 4 \cos(t)\)
8. \(y'' + 3y' + 2y = 4e^{-2t}\)

10. Use the integral definition of the Laplace transform to prove that $L[e^{at}f(t)] = F(s-a)$, where $f(s) = L[f(t)]$.

11. Use the integral definition of the Laplace transform to prove that $L[tf(t)] = (-1) \frac{dF}{ds}$. 
6.3 The Unit Step and Delta functions

It is common to encounter mechanical or electrical systems with various types of discontinuities. For an unbalanced engine on an airplane wing (essentially a forced mass-spring system), whenever the engine is stopped or started, there is a discontinuity in the forcing function. If the mass of a mass-spring system is hit with a hammer, there is a discontinuity in the velocity of mass at the moment of impact. In an electrical system, any time a switch is flipped, a discontinuity can occur in the input voltage. We will introduce two new functions that help to model such behavior.

The (Heaviside) Unit Step Function

**Definition** 6.3.1 The unit step function, $U(t)$, is defined as

$$U(t) = \begin{cases} 
0 & t < 0 \\
1 & t \geq 0
\end{cases}$$

It follows that the shifted unit step function $U(t - c)$ (it is shifted $c$ units to the right) would have the definition

$$U(t - c) = \begin{cases} 
0 & t < c \\
1 & t \geq c
\end{cases}$$

The graphs of $U(t)$ and $U(t - c)$ are shown in Figure 6.1.

![Figure 6.1](image)

**Figure 6.1:** The unit step function and shifted unit step function

Note: The unit step function is sometimes called the Heaviside function and denoted $H(t)$ instead of $U(t)$. 
We use the unit step function $U(t)$ and the shifted unit step function $U(t - c)$ to represent functions which are defined piecewise. For example, we can represent the function

$$f(t) =\begin{cases} 
0 & t < 0 \\
g(t) & 0 \leq t < a \\
h(t) & a \leq t < b \\
i(t) & b \leq t < \infty 
\end{cases}$$

(6.4)

as

$$f(t) = g(t)U(t) + (h(t) - g(t))U(t - a) + (i(t) - h(t))U(t - b)$$

(6.5)

Of course, the same principle applies for a piecewise function with more than, or fewer than, three pieces.

Note: For our purposes it will not matter how $f(t)$ is defined at the boundary points $0$, $a$, and $b$. This is because the Laplace transform of a function involves integration, and the value of an integral is not changed by altering the integrand at a single point. Thus we can still use the representation in Equation 6.5 to represent a function of the form given in Equation 6.4 even when the “less than or equal to” and “strictly less than” symbols are arranged differently (in fact, $f(t)$ need not even be defined at these points). We will use this representation for forcing functions in mass-spring systems and electrical systems.

**Example 6.3.1** Represent the function

$$f(t) =\begin{cases} 
0 & t < 0 \\
\sin(2t) & 0 < t < 5 \\
0 & 5 < t < \infty 
\end{cases}$$

using unit step functions. Also sketch a graph of $f(t)$. Note that $f(t)$ could represent, for example, a sinusoidal forcing function for a mass-spring system, which is turned on at $t = 0$ and then gets turned off at $t = 5$.

Referring to equations 6.4 and 6.5, with $g(t) = \sin(2t)$, $h(t) = 0$, and $a = 5$ (no $i(t)$ or $b$, since there are only two pieces) we get

$$f(t) = \sin(2t)U(t) + (0 - \sin(2t))U(t - 5)$$

There are several ways to plot this function. Your software may have a unit step function
(for example, in Maple it is called Heaviside(t)). If not, the function

\[ \text{sign}(t) = \begin{cases} 
-1 & t < 0 \\
1 & t > 0
\end{cases} \]

can be used to construct a unit step by writing

\[ U(t) \equiv \frac{1}{2} \cdot \text{sign}(t) + \frac{1}{2}. \]

The graph shown in Figure 6.2 was generated using Maple.

**The (Dirac) Delta Function**

Another function that is useful for representing certain types of forcing functions in mechanical and electrical systems is called the delta function, denoted \( \delta(t) \) (sometimes called the Dirac function or the Dirac delta function). This function has some rather unusual properties; in fact it is not really even a function at all in the usual sense. It is used to model a forcing function which represents a “hammer hit” on the mass of a mass-spring system, delivered at time \( t = 0 \). It can be thought of as a limit of functions \( \delta_\varepsilon(t) \), each of which is zero except on the time interval \( -\varepsilon < t < \varepsilon \), and each of which satisfies \( \int_{-\varepsilon}^{\varepsilon} \delta_\varepsilon(t)dt = 1 \).

Thus the functions \( \delta_\varepsilon(t) \) get narrower and taller as \( \varepsilon \) approaches 0, but always with area under the curve equal to 1. If we define \( \delta(t) = \lim_{\varepsilon \to 0} \delta_\varepsilon(t) \), then we get an infinitely tall, infinitely narrow function, which is 0 everywhere except at \( t = 0 \), and yet has area under
the curve equal to 1. In Figure 6.3 we show a sequence of such functions whose limit is \( \delta(t) \).

![Figure 6.3: A sequence of functions \( \delta_\varepsilon(t) \) whose limit as \( \varepsilon \to 0 \) is the delta function \( \delta(t) \)](image)

Since \( \delta(t) \) cannot be rigorously defined within the scope of this book (it can be rigorously defined using a branch of mathematics called “distribution theory” which is usually encountered in graduate level courses), we must be satisfied with describing its properties.

**Properties of \( \delta(t) \)**

The “function” \( \delta(t) \) satisfies the following properties:

1. \( \delta(t) = 0 \) for all \( t \) except \( t = 0 \).

2. \( \int_{t=a}^{t=b} \delta(t)\,dt = 1 \) for any \( a < 0 \) and any \( b > 0 \).

3. \( \int_{t=a}^{t=b} f(t)\delta(t)\,dt = f(0) \) for any continuous \( f(t) \) and any \( a < 0 \) and any \( b > 0 \).

As with the unit step function, the delta function can be shifted \( c \) units, so that the shifted delta function \( \delta(t - c) \) would represent a hammer hit on the mass of a mass-spring system at time \( t = c \). Its properties mirror those of the unshifted delta function.

**Properties of \( \delta(t - c) \)**

The “function” \( \delta(t - c) \) satisfies the following properties:

1. \( \delta(t - c) = 0 \) for all \( t \) except \( t = c \).
2. \[ \int_{t=a}^{t=b} \delta(t - c) \, dt = 1 \] for any \( a < c \) and any \( b > c \).

3. \[ \int_{t=a}^{t=b} f(t) \delta(t - c) \, dt = f(c) \] for any continuous \( f(t) \) and any \( a < c \) and any \( b > c \).

We provide an informal proof of Property 3, as we will need that property to find the Laplace transform of \( \delta(t) \).

**Proof of Property Three.** If we define

\[
\delta_\varepsilon(t) = \begin{cases} 
0 & t < -\varepsilon \\
\frac{1}{2\varepsilon} & -\varepsilon \leq t \leq \varepsilon \\
0 & t > \varepsilon 
\end{cases}
\]

then we get the sequence of functions pictured in Figure 6.3. Now, for any \( a < -\varepsilon \) and any \( b > \varepsilon \) we have \( \int_{t=a}^{t=b} \delta_\varepsilon(t) f(t) \, dt = \int_{t=\varepsilon}^{t=-\varepsilon} \delta_\varepsilon(t) f(t) \, dt = \int_{t=\varepsilon}^{t=-\varepsilon} \frac{1}{2\varepsilon} f(t) \, dt = \frac{1}{2\varepsilon} \int_{t=-\varepsilon}^{t=\varepsilon} f(t) \, dt \). We can represent the integral of a continuous function over an interval as its average value on that interval, denoted \( f_{\text{avg}} \), multiplied by the length of the interval. Thus \( \int_{t=-\varepsilon}^{t=\varepsilon} f(t) \, dt = f_{\text{avg}} \cdot 2\varepsilon \) and so \( \int_{t=a}^{t=b} \delta_\varepsilon(t) f(t) \, dt = \frac{1}{2\varepsilon} f_{\text{avg}} \cdot 2\varepsilon = f_{\text{avg}} \). Now, as \( \varepsilon \to 0 \), \( \delta_\varepsilon(t) \to \delta(t) \), and \( f_{\text{avg}} \to f(0) \) (the average value of a continuous function over a very small interval is about equal to the value of the function at the midpoint of the interval). Thus \( \int_{t=a}^{t=b} \delta(t) f(t) \, dt = f(0) \). The proof that \( \int_{t=a}^{t=b} \delta(t - c) f(t) \, dt = f(c) \) is similar.

**Physical Interpretation for \( \delta(t) \):** The delta function will only be used in this text as part of a forcing function for a driven mass-spring system. In that context, \( \delta(t - c) \) represents a unit impulse applied at time \( t = c \). A unit impulse acting on a 1 kilogram mass increases the velocity instantaneously by 1 meter per second. A unit impulse acting on an \( m \) kilogram mass increases the velocity instantaneously by \( \frac{1}{m} \) meters per second. Finally, \( N \cdot \delta(t - c) \) represents an impulse of \( N \) units applied at time \( t = c \), meaning it increases the velocity of an \( m \) kilogram mass instantaneously by \( \frac{N}{m} \) meters per second.

**Laplace Transforms of \( U(t - c) \) and \( \delta(t - c) \)**

In order to solve differential equations that involve \( U(t - c) \) and \( \delta(t - c) \) we need to find their Laplace transforms and add them to our list of Laplace transform formulas.
Theorem 6.5 The Laplace transforms of $U(t-c)$ and $\delta(t-c)$ are given in the table below.

<table>
<thead>
<tr>
<th>formula</th>
<th>function</th>
<th>Laplace transform</th>
</tr>
</thead>
<tbody>
<tr>
<td>17</td>
<td>$U(t-c)$</td>
<td>$\frac{1}{s}e^{-sc}$</td>
</tr>
<tr>
<td>18</td>
<td>$\delta(t-c)$</td>
<td>$e^{-sc}$</td>
</tr>
</tbody>
</table>

Table 6.5: Laplace transforms of $U(t-c)$ and $\delta(t-c)$

Proof: See the exercises at the end of this section.

We need one more Laplace transform formula before we can start solving differential equations that involve $U(t-c)$ and $\delta(t-c)$. It is one of the “general” properties that apply to any function $f(t)$, such as the properties given in formulas 10 and 11 from Theorem 6.4.

Theorem 6.6 The Laplace transform of $f(t-c)U(t-c)$ is given in the table below, where $F(s)$ is the Laplace transform of the function $f(t)$, and $c$ is a real constant.

<table>
<thead>
<tr>
<th>formula</th>
<th>function</th>
<th>Laplace transform</th>
</tr>
</thead>
<tbody>
<tr>
<td>19</td>
<td>$f(t-c)U(t-c)$</td>
<td>$e^{-cs}F(s)$</td>
</tr>
</tbody>
</table>

Table 6.6: Laplace transform of $f(t-c)U(t-c)$

Proof:

$$L[f(t-c)U(t-c)] = \int_0^\infty f(t-c)U(t-c)e^{-st}dt = \int_c^\infty f(t-c)e^{-st}dt;$$

then, substituting $u = t - c$,

$$= \int_0^\infty f(u)e^{-s(u+c)}du = e^{-sc} \int_0^\infty f(u)e^{-su}du = e^{-cs}F(s).$$

Note: The graph of the function $f(t-c)U(t-c)$ is just the graph of $f(t)$, shifted $c$ units to the right, and set equal to zero up to time $t = c$.

Formula 19 of Theorem 6.6 is especially useful in finding the inverse Laplace transform of functions involving an exponential function $e^{-cs}$.

Example 6.3.2 Use Theorem 6.6 to find $L^{-1}\left[e^{-2s}\frac{s}{s^2+9}\right]$ and $L^{-1}\left[e^{-5s}\frac{4}{(s+1)^2+16}\right]$.

For the first inverse Laplace transform problem, we let $F(s)$ be $\frac{s}{s^2+9}$. We should recognize $F(s)$ as fitting the right-hand side of formula 3 with $b = 3$, so we have $f(t) = L^{-1}[F(s)] = \ldots$
\[
\cos(3t) \text{ (left-hand side of formula 3). The } e^{-2s} \text{ part means we need to use formula } 19 \text{ with } c = 2. \text{ Since } f(t - 2) = \cos(3(t - 2)) \text{ we have }
\]
\[
L^{-1}\left[\frac{e^{-2s}}{s^2 + 9}\right] = L^{-1}[e^{-2s}F(s)] = f(t - 2)U(t - 2)
\]
\[
= \cos(3(t - 2))U(t - 2).
\]

For the second inverse problem, we use \( F(s) = \frac{4}{(s+1)^2+16} \), which fits the right-hand side of formula 13 with \( a = -1 \), and \( b = 4 \); so from the left-hand side of formula 13 we get \( f(t) = e^{-1-t}\sin(4t) \). Now using formula 19 we have \( c = 5 \) (from the \(-5s\) term), and with \( f(t - 5) = e^{-(t-5)}\sin(4(t - 5)) \) we get
\[
L^{-1}\left[e^{-5s}\frac{4}{(s+1)^2+9}\right] = L^{-1}[e^{-5s}F(s)] = f(t - 5)U(t - 5)
\]
\[
= e^{-(t-5)}\sin(4(t - 5))U(t - 5).
\]

Let’s solve some differential equations!

**Example 6.3.3** Solve the initial value problem \( y'' + 4y = 10\sin(t - 10)U(t - 10) \), \( y(0) = 3 \), \( y'(0) = 0 \) using Laplace transforms. Sketch both the input (forcing function) and the response \( y(t) \). Discuss the problem and its solution in the context of a mass-spring problem.

For the right-hand side we can use formula 19 with \( c = 10 \). Taking the Laplace transform of both sides we have
\[
(s^2Y - sy(0) - y'(0)) + 4Y = 10e^{-s10} \frac{1}{s^2 + 1}
\]
so that after replacing \( y(0) \) and \( y'(0) \) with their numeric values, and isolating the \( Y \) terms on the left we get
\[
s^2Y + 4Y = 10e^{-s10} \frac{1}{s^2 + 1} + 3s
\]
and after factoring out \( Y \) and dividing by \( s^2 + 4 \) we get
\[
Y = 10e^{-10s} \frac{1}{(s^2 + 4)(s^2 + 1)} + \frac{3s}{(s^2 + 4)}.
\]
The term $\frac{1}{(s^2+4)(s^2+1)}$ needs to be expanded (using partial fractions):

$$\frac{1}{(s^2+4)(s^2+1)} = \frac{1}{3(s^2+1)} - \frac{1}{3(s^2+4)}$$

so that

$$Y(s) = 10e^{-10s} \left( \frac{1}{3(s^2+1)} - \frac{1}{3(s^2+4)} \right) + \frac{3s}{(s^2+4)}$$

Taking the inverse Laplace transform of both sides yields

$$y(t) = L^{-1} \left[ \frac{10}{3} e^{-10s} \left( \frac{1}{s^2+1} \right) - \frac{10}{3} \frac{e^{-10s}}{s^2+4} + \frac{3s}{(s^2+4)} \right]$$

For the first two terms we combine formula 4 (letting $F(s) = \frac{1}{s^2+1}$, $f(t) = \sin(t)$ for the first term and $F(s) = \frac{2}{s^2+1}$, $f(t) = \sin(2t)$ for the second term) with formula 19 ($c = 10$). This is similar to what we did in Example 6.3.2. The third term is just formula 4 alone. We end up with

$$y(t) = \frac{10}{3} \sin (t-10) U(t-10) - \frac{5}{3} \sin (2(t-10)) U(t-10) + 3 \cos (2t)$$

Graphs of the forcing function $f(t) = 10\sin(t-10) U(t-10)$ and the response $y(t) = \frac{10}{3} \sin (t-10) U(t-10) - \frac{5}{3} \sin (2(t-10)) U(t-10) + 3 \cos (2t)$ are shown in Figure 6.4.
The differential equation \( y'' + 4y = 10 \sin(t - 10)U(t - 10) \) models a mass-spring system with mass 1 kilogram, no damping, and spring constant 4 newtons per meter. The mass is displaced 3 meters in the positive direction \( (y(0) = 3) \) and released \( (y'(0) = 0) \). A sinusoidal forcing function with frequency \( \frac{1}{2\pi} \) and amplitude 10 starts after 10 seconds.

The solution

\[
y(t) = \frac{10}{3} \sin(t - 10)U(t - 10) - \frac{5}{3} \sin(2(t - 10))U(t - 10) + 3 \cos(2t)
\]

shows that there is an initial response of \( 3 \cos(2t) \) which defines the motion of the system prior to \( t = 10 \) seconds. After the forcing function “kicks in” the response consists of the three terms \( \frac{10}{3} \sin(t - 10) - \frac{5}{3} \sin(2(t - 10)) + 3 \cos(2t) \). The first term, \( \frac{10}{3} \sin(t - 10) \), represents a wave with frequency \( \frac{1}{2\pi} \) and the second two terms, \( -\frac{5}{3} \sin(2(t - 10)) + 3 \cos(2t) \), combine to form a wave of frequency \( \frac{2}{2\pi} = \frac{1}{\pi} \) (in Section 3.2 we discussed how to combine trig functions with the same frequency to get a single trig function with that frequency).

Thus the response \( y(t) \) after \( t = 10 \) consists of the interaction of two waves with different frequencies, producing the results in Figure 6.4.

Our last example combines the unit step and delta functions.

**Example 6.3.4** A mass of 1 kilogram is suspended on a spring with spring constant 145 newtons per meter, and a damping constant of 2 newtons per meter per second. The mass is hit from above with a hammer giving it an initial velocity of 2 meters per second downward (at \( t = 0 \) seconds). A rocket is fired at the mass from below, and impacts the mass at \( t = 1 \) second. The impact results in an impulse of 2 units (modeled by \( 2\delta(t - 1) \)) and a decaying exponential force which starts at 5 newtons and decays at an instantaneous rate of 20% per second (modeled by \( 5e^{-0.2(t-1)}U(t - 1) \)).

Write the differential equation which describes this process, and solve it for \( y(t) \). Graph both \( y(t) \) and \( y'(t) \) as functions of \( t \). Describe what happens in these two graphs at \( t = 1 \) and after \( t = 1 \).
Our mass-spring system has $m = 1$, $c = 2$, and $k = 145$, so the nonhomogeneous equation (driven system) is

$$y'' + 2y' + 145y = 2\delta(t - 1) + 5e^{-0.2(t-1)}U(t - 1).$$

The initial conditions are $y(0) = 0$ and $y'(0) = -2$. Taking the Laplace transform of both sides of the differential equation, and using linearity, we get

$$L[y''] + 2L[y'] + 145L[y] = 2L[\delta(t - 1)] + 5L[e^{-0.2(t-1)}U(t - 1)] \quad (6.6)$$

Using formulas 8 and 9 from section 6.1 for the left-hand side of equation 6.6, and Table 6.5 for the first term on the right-hand side, and Table 6.6 (with $f(t) = e^{-0.2t}$, $F(s) = \frac{1}{s+0.2}$, $c = 1$) for the second term on the right-hand side we get

$$(s^2Y - s \cdot 0 - (-2)) + 2(sY - 0) + 145Y = 2e^{-1}s + 5\left(e^{-1}s \cdot \frac{1}{s + 0.2}\right)$$

Now collect $Y$ terms on the left and factor out $Y$:

$$(s^2 + 2s + 145)Y = -2 + 2e^{-s} + 5e^{-s} \cdot \frac{1}{s + 0.2}$$

and then divide by $s^2 + 2s + 145$ to get

$$Y = -\frac{2}{s^2 + 2s + 145} + \frac{2e^{-s}}{s^2 + 2s + 145} + \frac{5e^{-s}}{(s^2 + 2s + 145)(s + 0.2)} \quad (6.7)$$

For the first two terms in equation 6.7 we complete the square in the denominator

$$s^2 + 2s + 145 = s^2 + 2s + 1 - 1 + 145 = (s + 1)^2 + 144$$

and for the last term in equation (6.7) we use partial fractions first, and then complete the square (turning decimal numbers into fractions is often helpful when using computer algebra):

$$\frac{1}{(s^2 + 2s + 145)(s + \frac{1}{5})} = \left(\frac{1}{3616}\right) \cdot \left(\frac{125}{5s+1}\right) + \left(\frac{1}{3616}\right) \cdot \left(\frac{-25s - 45}{2s + s^2 + 145}\right)$$

$$= \left(\frac{1}{3616}\right) \cdot \left(\frac{1}{5}\right) \cdot \left(\frac{125}{s + \frac{1}{5}}\right) + \left(\frac{1}{3616}\right) \cdot \left(\frac{-25s - 45}{(s + 1)^2 + 144}\right)$$
Equation (6.7) now becomes
\[
Y = -2 \frac{1}{(s+1)^2 + 144} + 2e^{-s} \frac{1}{(s+1)^2 + 144}
+ 5e^{-s} \left( \frac{1}{3616} \cdot \frac{1}{5} \cdot \frac{125}{s+\frac{1}{5}} + \frac{1}{3616} \cdot \left( \frac{-25s - 45}{(s+1)^2 + 144} \right) \right)
= -\frac{2}{12} \cdot \frac{12}{(s+1)^2 + 144} + \frac{2}{12} \cdot e^{-s} \frac{12}{(s+1)^2 + 144}
+ \frac{125}{3616} \cdot e^{-s} \frac{1}{s+\frac{1}{5}} + \frac{5}{3616} \cdot e^{-s} \frac{-25s - 45}{(s+1)^2 + 144}
\]
(6.8)

after multiplying out and using the standard “multiply and divide” trick on the first two terms. The first three terms are now set up for the inverse Laplace transformation, but the last needs some work. We need to split it up and use the “add and subtract” trick:

\[
\frac{-25s - 45}{(s+1)^2 + 144} = -25 \frac{s+1-1}{(s+1)^2 + 144} - 45 \frac{1}{(s+1)^2 + 144}
= -25 \frac{s+1}{(s+1)^2 + 144} + 25 \frac{1}{(s+1)^2 + 144} - 45 \frac{1}{(s+1)^2 + 144}
= -25 \frac{s+1}{(s+1)^2 + 144} - 20 \frac{12}{(s+1)^2 + 144}
\]

Substituting this last expression into equation 6.8 we get
\[
Y = -\frac{2}{12} \cdot \frac{12}{(s+1)^2 + 144} + \frac{2}{12} \cdot e^{-s} \frac{12}{(s+1)^2 + 144} + \frac{125}{3616} \cdot e^{-s} \frac{1}{s+\frac{1}{5}}
+ \frac{5}{3616} \cdot e^{-s} \left( -25 \frac{s+1}{(s+1)^2 + 144} - 20 \frac{12}{(s+1)^2 + 144} \right)
= -\frac{1}{6} \cdot \frac{12}{(s+1)^2 + 144} + \frac{1}{6} \cdot e^{-s} \frac{12}{(s+1)^2 + 144} + \frac{125}{3616} \cdot e^{-s} \frac{1}{s+\frac{1}{5}}
- \frac{125}{3616} \cdot e^{-s} \frac{s+1}{(s+1)^2 + 144} - \frac{25}{10848} \cdot e^{-s} \frac{12}{(s+1)^2 + 144}
\]

To find the inverse Laplace transform we use formulas 2, 12 and 13, combined with formula
The Unit Step and Delta functions

19 when necessary. We get

\[ y(t) = L^{-1}[Y] = -\frac{1}{6}L^{-1}\left[\frac{12}{(s+1)^2 + 144}\right] + \frac{1}{6}L^{-1}\left[\frac{2s}{(s+1)^2 + 144}\right] + \frac{125}{3616}L^{-1}\left[\frac{1}{s + \frac{1}{5}}\right] - \frac{125}{3616}L^{-1}\left[\frac{s + 1}{(s+1)^2 + 144}\right] - \frac{25}{10848}L^{-1}\left[\frac{12}{(s+1)^2 + 144}\right] \]

\[ = -\frac{1}{6}e^{-t}\sin(12t) + \frac{1}{6}e^{-(t-1)}\sin(12(t-1))U(t-1) + \frac{125}{3616}e^{-\frac{1}{5}(t-1)}U(t-1) - \frac{125}{3616}e^{-(t-1)}\cos(12(t-1))U(t-1) - \frac{25}{10848}e^{-(t-1)}\sin(12(t-1))U(t-1). \]

It is standard to combine like terms, either at this point, or before the inverse Laplace transform is taken. The final equation after combining terms is

\[ y(t) = -\frac{1}{6}e^{-t}\sin(12t) + \frac{125}{3616}e^{-\frac{1}{5}(t-1)}U(t-1) - \frac{125}{3616}e^{-(t-1)}\cos(12(t-1))U(t-1) + \frac{1783}{10848}e^{-(t-1)}\sin(12(t-1))U(t-1). \]

The graphs of \(y(t)\) and \(y'(t)\) are shown in Figure 6.5.

---

**Figure 6.5:** Position \(y(t)\) (left) and Velocity \(y'(t)\) (right)

One can see from the graph of \(y(t)\) that at \(t = 1\), at the time of impact of the rocket, the mass reverses direction from “down” to “up” creating a sharp corner in the graph. The graph of \(y'(t)\) shows a discontinuity at this point, flipping instantaneously from negative to positive. After \(t = 1\), the rocket continues to fire, but with exponentially decreasing force. This causes the oscillations to become raised (they are no longer centered on \(y = 0\), but
the center of the oscillations is gradually coming back down towards \( y = 0 \) as the rocket force dies out.

We conclude this section with a list of all the Laplace transform formulas that we have developed up to this point.
# TABLE OF LAPLACE TRANSFORMS

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Exercises In problems 1-6, represent the piecewise-defined function \( f(t) \) using the unit step function \( U(t) \). Graph the function \( f(t) \) using an appropriate interval. Is \( f(t) \) continuous?

1. \( f(t) = \begin{cases} 0 & t < \pi \\ \sin(2(t - \pi)) & t \geq \pi \end{cases} \)

2. \( f(t) = \begin{cases} 0 & t < 2 \\ (t - 2)^3 + 4 & 2 \leq t < \infty \end{cases} \)

3. \( f(t) = \begin{cases} 0 & t < 10 \\ e^{-0.2(t-10)} \cos(t-10) & t \geq 10 \end{cases} \)

4. \( f(t) = \begin{cases} \sin(2t) & 0 \leq t < \pi \\ 0 & \pi \leq t < \infty \end{cases} \)

5. \( f(t) = \begin{cases} 2e^{-t} & 0 \leq t < 1 \\ 0 & 1 \leq t < \infty \end{cases} \)

6. \( f(t) = \begin{cases} e^{-0.2t} \cos(t) & 0 \leq t < 10 \\ e^{-0.2(t-10)} \cos(t-10) & 10 \leq t < \infty \end{cases} \)

For problems 7-12, find each inverse Laplace transform using formula 19 in conjunction with another appropriate Laplace transform formula.

7. \( L^{-1}\left[\frac{e^{-3s}}{s-4}\right] \)

8. \( L^{-1}\left[\frac{e^{2s}3}{s}\right] \)

9. \( L^{-1}\left[\frac{e^{-3s}}{s^3}\right] \)

10. \( L^{-1}\left[\frac{e^{-3s}-s}{(s^2+4)^2}\right] \)

11. \( L^{-1}\left[\frac{e^{-3s}}{(s-a)^3}\right] \)

12. \( L^{-1}\left[\frac{e^{-3s}}{(s-1)^2+16}\right] \)

For problems 13-20, solve the initial value problem using Laplace transforms. Graph the input (forcing function) and the solution, and interpret the differential equation, the initial conditions, and the solution as a model for a mass-spring system.
13. \( y'' + 4y = f(t), \ y(0) = 1, \ y'(0) = 0 \), where \( f(t) = \begin{cases} 0 & t < 5 \\ 3\cos(t-5) & 5 \leq t < \infty \end{cases} \)

14. \( y'' + 9y = f(t), \ y(0) = 1, \ y'(0) = 0 \), where \( f(t) = \begin{cases} 0 & t < 2 \\ (t-2) & 2 \leq t < \infty \end{cases} \)

15. \( y'' + 4y' + 8y = f(t), \ y(0) = 1, \ y'(0) = 0 \), where \( f(t) = \begin{cases} 0 & t < 2 \\ (t-2)^2 & 2 \leq t < \infty \end{cases} \)

16. \( y'' + 5y' + 6y = f(t), \ y(0) = 0, \ y'(0) = 1 \), where \( f(t) = \begin{cases} 0 & t < 3 \\ \sin(t-3) & 3 \leq t < \infty \end{cases} \)

17. \( y'' + 3y' + 2y = f(t), \ y(0) = 0, \ y'(0) = 0 \), where \( f(t) = \begin{cases} 2e^{-3t} & 0 \leq t < 1 \\ 0 & 1 \leq t < \infty \end{cases} \)

Hint: \( 2e^{-3t} = 2e^{-3(t-1)+1} = 2e^{-3(t-1)} - 3 = 2e^{-3(t-1)}e^{-3} \)

18. \( y'' + 2y' + 2y = f(t), \ y(0) = 0, \ y'(0) = 0 \), where \( f(t) = \begin{cases} \cos(t) & 0 \leq t < 5 \\ 0 & 5 \leq t < \infty \end{cases} \)

Hint: \( \cos(t) = \cos(t-5+5) = \cos(t-5)\cos(5) - \sin(t-5)\sin(5) \)

19. \( y'' + 4y = 3\delta(t-3), \ y(0) = 1, \ y'(0) = 0 \)

20. \( y'' + 3y' + 2y = 2\delta(t-3), \ y(0) = 1, \ y'(0) = 0 \)

For problems 21 and 22, develop a model for the mass-spring system, and solve for the position \( y(t) \). Graph \( y(t) \) and \( y'(t) \) and discuss the graphs in the context of the mass-spring system.

21. Mass \( m = 1 \) kilogram, no damping, spring constant \( k = 9 \) newtons per meter. The mass is displaced 1 meter in the positive direction and released. At \( t = 5 \) seconds a sinusoidal forcing function begins, with frequency \( \frac{1}{2\pi} \) and amplitude 2 (modeled by \( 2 \sin(t-5) \)).

22. Mass \( m = 1 \) kilogram, damping constant \( c = 5 \) newtons per meter per second, spring constant \( k = 6 \) newtons per meter. The mass is hit with a hammer giving it an initial velocity of 2 meters per second. At \( t = 2 \) seconds the mass is hit with a hammer again, imparting an impulse of 3 units in the downward direction.
23. Use the definition of the Laplace transform to prove that \( L[U(t - c)] = \frac{1}{s}e^{-sc} \). Remember that \( U(t - c) \) is zero except when \( t > c \).

24. Use Property 3 of \( \delta(t - c) \) to prove that \( L[\delta(t - c)] = e^{-sc} \).
6.4 Convolution and Circuits

Definition 6.4.1 The convolution of two functions $f(t)$ and $g(t)$ is denoted $(f * g)(t)$ and is defined to be

$$(f * g)(t) = \int_{u=0}^{u=t} f(u)g(t-u)du$$

Theorem 6.7 $(f * g)(t) = (g * f)(t)$

Theorem 6.8 $L[(f * g)(t)] = F(s)G(s)$

Note: We will most often use this theorem in the form $L^{-1}[F(s)G(s)] = (f * g)(t)$.

Example 6.4.1 Find $L^{-1}\left[\frac{1}{(s^2+4)^2}\right]$ two different ways; first without convolution, then again using convolution.

To do this problem without convolution we can employ Laplace formula 14, which is $L^{-1}\left[\frac{s^2-b^2}{(s^2+b^2)^2}\right] = t \cos bt$. We use the “multiply and divide” trick followed by the “add and subtract” trick. We have

$$\frac{1}{(s^2+4)^2} = \frac{1}{8} \frac{8}{(s^2+4)^2} = \frac{1}{8} \frac{1}{(s^2+4)^2} = \frac{1}{8} \frac{4 - s^2}{(s^2+4)^2} + \frac{1}{8} \frac{s^2 + 4}{(s^2+4)^2}$$

$$= \frac{1}{8} \frac{s^2 - 4}{(s^2+4)^2} + \frac{1}{8} \frac{1}{s^2 + 4}$$

from which we get

$$L^{-1}\left[\frac{1}{(s^2+4)^2}\right] = -\frac{1}{8} L^{-1}\left[\frac{s^2 - 4}{(s^2+4)^2}\right] + \frac{1}{16} L^{-1}\left[\frac{2}{s^2 + 4}\right]$$

$$= -\frac{1}{8} t \cos(2t) + \frac{1}{16} \sin(2t)$$

With convolution we use $F(s) = \frac{1}{s^2+1}$ and also $G(s) = \frac{1}{s^2+4}$. Thus we get $f(t) = \ldots$
\[ L^{-1}[F(s)] = L^{-1}\left[\frac{1}{\frac{1}{2} s^2 + 4}\right] = \frac{1}{2} \sin(2t) \text{ and so also } g(t) = \frac{1}{2} \sin(2t). \] Thus

\[
L^{-1}\left[\frac{1}{(s^2 + 4)^2}\right] = L^{-1}[F(s)G(s)] = (f * g)(t)
\]

\[
= \int_{u=0}^{u=t} \frac{1}{2} \sin(2u) \frac{1}{2} \sin(2(t - u)) du
\]

\[
= \frac{1}{16} \sin(2t) - \frac{1}{8} t \cos(2t)
\]

as we got above (using computer algebra to do the integral). We note that the convolution method is conceptually simpler, but that the integral that results may not be simple without computer algebra. In the exercises at the end of this section you are asked to evaluate the integral above without computer algebra, given some hints.

**Electrical Circuits**

In Section 1.4 we develop in some detail the rules for developing differential equations that correspond to various electrical circuits. Below we give an abbreviated version, which should be sufficient for this section.

In electrical circuits that contain resistors, capacitors and inductors, the voltage drop across any one of these components can be related to the current flowing through that component by simple linear relationships. If we let \( i(t) \) represent the current, and \( v(t) \) the voltage drop across a given component, then we have the following proportionality relations:

- **Drop across resistor:** \( v = Ri \)
- **Drop across capacitors:** \( C \frac{dv}{dt} = i \)
- **Drop across inductor:** \( v = L \frac{di}{dt} \)

The proportionality constants \( R, C, \) and \( L, \) are called the resistance, the capacitance, and the inductance respectively. One of the primary means of creating differential equation models for circuits is through the use of Kirchoff’s Second Law, which states that the sum of all of the voltage drops around a closed circuit must equal the voltage gain (due to a voltage source such as a battery).

A circuit with a resistor, a capacitor, and an inductor in series is called an RLC circuit (see Figure 6.6).
Thus if we have a resistor with voltage drop $v_r$, a capacitor with voltage drop $v_c$, an inductor with voltage drop $v_l$ and a voltage source $v_s$ we would have $v_r + v_c + v_l = v_s$. If we consider these voltages to be time-dependent, then we can take the derivative of both sides to get $rac{dv_r}{dt} + rac{dv_c}{dt} + rac{dv_l}{dt} = rac{dv_s}{dt}$. Combining this equation with the voltage drop proportionality above we get $R \frac{di}{dt} + \frac{1}{C} i + L \frac{d^2i}{dt^2} = \frac{dv_s}{dt}$, or rearranging

$$L \frac{d^2i}{dt^2} + R \frac{di}{dt} + \frac{1}{C} i = \frac{dv_s}{dt} \quad (6.9)$$

which is the second order differential equation that governs an RLC circuit.

Other common circuits are ones where there is just a resistor and a capacitor ($RC$ circuit) or just a resistor and an inductor ($RL$ circuit). The differential equations for these cases are first order differential equations, which can be obtained by dropping out the appropriate term. Also, the current $i$ is defined as the derivative of the charge $q$. Using $i = \frac{dq}{dt}$ allows us to write our differential equations in terms of $q$ when that is more convenient.

**Units**: Resistance $R$ is measured in Ohms, capacitance $C$ is measured in Farads, inductance $L$ is measured in Henries, voltage is measured in volts ($V$), and current $i$ is measured in amperes.

**Example 6.4.2** An $RLC$ circuit has a capacitance of $\frac{1}{2}$ Farad, a resistance of 2 Ohms, an inductance of 1 Henry, and a voltage source of $\sin(t)$ volts. Find the long-term response of the system (that is, find $i(t)$ for large $t$). What is the amplitude of the long-term response?

We have $C = \frac{1}{2}$, $R = 2$, $L = 1$, and $v_s = \sin(t)$ and hence $\frac{dv_s}{dt} = \cos(t)$. Employing equation 6.9 we get

$$L \frac{d^2i}{dt^2} + 2 \frac{di}{dt} + 2i = \cos(t). \quad (6.10)$$

Even though we are going to employ Laplace transforms to solve this differential equation, we will first use the techniques of Chapter 3 to observe that the characteristic equation is
$r^2 + 2r + 2 = 0$, which has solutions $r = -1 \pm i$. Thus the homogeneous solution would be $i_h = C_1e^{-t}\cos t + C_2e^{-t}\sin t$. As $t \to \infty$ we have $i_h \to 0$, so that the long-term response consists of only the particular solution $i_p$. An important consequence of this fact is that the initial conditions $i(0)$ and $i'(0)$ have no affect on the long term behavior of the system. Hence to make the algebra of Laplace transforms simpler, we can assume $i(0) = 0$ and $i'(0) = 0$.

Taking the transform of both sides of equation 6.10 we get

$$s^2 I(s) + 2sI(s) + 2I(s) = \frac{s}{s^2 + 1}$$

and then solving for $I(s)$ we get

$$I(s) = \frac{s}{s^2 + 1 \cdot s^2 + 2s + 2}.$$  \hspace{1cm} (6.11)

At this point we could use the techniques of Sections 6.1 and 6.2 (expand, complete the square), but instead we choose convolution. With $F(s) = \frac{s}{s^2 + 1}$ and $G(s) = \frac{1}{s^2 + 2s + 2} = \frac{1}{(s+1)^2 + 1}$ we have $f(t) = \cos t$ and $g(t) = e^{-t}\sin t$. Thus

$$i(t) = (f * g)(t) = (g * f)(t) = \int_{u=0}^{u=t} g(u)f(t-u)du$$

$$= \int_{u=0}^{u=t} (e^{-u}\sin u) \cos(t-u)du$$

$$= \frac{1}{5} \cos(t) + \frac{2}{5} \sin(t) - \frac{1}{5} e^{-t}\cos t - \frac{3}{5} e^{-t}\sin t,$$

using computer algebra to do the integral. Thus the long-term response would be given by

$$i_p = \frac{1}{5} \cos(t) + \frac{2}{5} \sin(t).$$

Finally, we can combine sine and cosine terms as shown in Section 3.3:

$$i_p(t) = \frac{2}{5} \sin(t) + \frac{1}{5} \cos(t) = D\cos(t - \phi).$$

We have $D^2 = \left(\frac{2}{5}\right)^2 + \left(\frac{1}{5}\right)^2 = \frac{1}{5}$ and $\tan(\phi) = \frac{2/5}{1/5} = 2$. With $D = \sqrt{\frac{1}{5}}$ we must choose
\( \phi = \arctan(2) \approx 0.352 \pi \) (show why \( \phi = \arctan(2) + \pi \) does not work). Thus

\[
i_p(t) \approx \sqrt{\frac{1}{5}} \cos(t - 0.352 \pi).
\]

The amplitude of the long-term response is \( \sqrt{\frac{1}{5}} \approx 0.44721 \).

An important concept that we need to take away from Example 6.4.2 is that whenever there is non-zero resistance in an RLC circuit, and we are only interested in the long-term output, we can set the initial conditions to zero. We will take that course for the rest of this section.

### Transfer Functions and the Frequency Domain

Equation 6.11 from Example 6.4.2 provides another important concept that we can apply in general to electrical circuits (and mechanical systems as well) for which the initial conditions are zero. When we solve an initial value problem of the form \( ay'' + by' + cy = f(t) \), \( y(0) = 0 \), \( y'(0) = 0 \) (second-order linear), or of the form \( ay' + by = f(t) \), \( y(0) = 0 \) (first-order linear), we interpret the function \( f(t) \) as the input and the solution \( y(t) \) as the output (or the response).

**Definition 6.4.2** For a nonhomogeneous linear differential equation with input (forcing function) \( f(t) \) and output (solution) \( y(t) \), and zero initial conditions, the ratio of the Laplace transform of the output \( Y(s) \) to the Laplace transform of the input \( F(s) \) is called the transfer function. We will denote the transfer function by \( H(s) = \frac{Y(s)}{F(s)} \).

Since \( Y(s) = H(s)F(s) \), we have \( y(t) = L^{-1}[Y(s)] = L^{-1}[H(s)F(s)] = (h * f)(t) \). Both \( y \) and \( f \) are functions of \( t \) (time), hence when we deal with these functions we say that we are working in the time domain. The functions \( Y \) and \( F \) are functions of \( s \), so that when we work with these functions we say we are working in the frequency domain (also called the \( s \)-plane). Because multiplication is a simpler operation than convolution, many electrical engineers prefer working in the frequency domain.

In Example 6.4.2, the input is \( f(t) = \cos t \) and the Laplace transform of the input is \( F(s) = \frac{s}{s^2 + 1} \). The Laplace transform of the output is \( I(s) \). From equation 6.11, we see that the transfer function for that example is given by \( H(s) = \frac{I(s)}{F(s)} = \frac{1}{s^2 + 2s + 2} \).
Example 6.4.3 An RLC circuit has in input given by \( F(s) = \frac{5s}{s^2 + 9} \) and transfer function given by \( H(s) = \frac{1}{s^2 + 2s + 5} \). Find the output in both the frequency domain (that is \( Y(s) \)) and in the time domain (that is \( y(t) \)), and find the long-term amplitude in the time domain. Also, find the initial value problem (differential equation plus initial conditions) corresponding to the given input and transfer function, and state the capacitance, resistance, and inductance of the circuit.

The output in the frequency domain is just the product of the input and the transfer function, thus
\[
Y(s) = H(s)F(s) = \frac{1}{s^2 + 2s + 5} \cdot \frac{5s}{s^2 + 9}.
\]

To find the output in the time domain we use convolution, along with computer algebra to evaluate the integral. First we need to transform \( F(s) \) and \( H(s) \) to the time domain. We have
\[
f(t) = L^{-1}[F(s)] = L^{-1}\left[\frac{5s}{s^2 + 9}\right] = 5 \cos(3t)
\]
and
\[
h(t) = L^{-1}[H(s)] = L^{-1}\left[\frac{1}{s^2 + 2s + 5}\right] = L^{-1}\left[\frac{1}{s^2 + 2s + 1 + 4}\right] = L^{-1}\left[\frac{1}{(s + 1)^2 + 4}\right] = \frac{1}{2}e^{-t} \sin(2t).
\]

Thus
\[
y(t) = L^{-1}[Y(s)] = L^{-1}[H(s)F(s)] = (h \ast f)(t) = \int_{u=0}^{u=t} h(t-u)f(u)\,du
\]
\[
= \int_{0}^{t} \frac{1}{2}e^{-(t-u)} \sin(2(t-u)) \cdot 5 \cos(3u)\,du
\]
\[
= \frac{15}{26} \sin 3t - \frac{5}{13} \cos 3t + \frac{5}{13} (\cos 2t) e^{-t} - \frac{35}{52} (\sin 2t) e^{-t}
\]
(Note : The form of \( y(t) \) will vary between different computer algebra systems).

In the long term (that is, as \( t \to \infty \)), \( y(t) \) approaches the steady-state solution \( \frac{15}{26} \sin 3t - \frac{5}{13} \cos 3t \). The amplitude of the steady-state solution would be \( \sqrt{(\frac{15}{26})^2 + (\frac{5}{13})^2} = \frac{5}{26}\sqrt{13} \approx 0.69338 \).

To determine the differential equation, we write the equation \( Y(s) = H(s)F(s) \) as \( Y(s) \frac{1}{H(s)} = \)}
\[ F(s) \text{ which becomes} \]
\[ Y(s)(s^2 + 2s + 5) = 5 \frac{s}{s^2 + 9} \]
or equivalently
\[ s^2Y(s) + 2sY(s) + 5Y(s) = 5\frac{s}{s^2 + 9} \]

We can see that the above equation is the result of taking the Laplace transform of both sides of the differential equation

\[ y''(t) + 2y'(t) + 5y(t) = 5 \cos(3t) \]

if we take the initial conditions to be \( y(0) = 0 \) and \( y'(0) = 0 \) (check this). Thus \( L = 1 \) Henry, \( R = 2 \) Ohms, and \( C = \frac{1}{5} \) Farads.

**Filters and the Response Curve**

One of the points of Example 6.4.3 is that all of the important information about a circuit is contained in the transfer function \( H(s) \) and the input \( F(s) \). If the input is not specified, then the transfer function tells the whole story; it specifies how to get from the input to the output (in either the time or frequency domain).

One important property of an electrical circuit is how the frequency of the input affects the amplitude of the output (in the long term). In Example 6.4.3, the frequency of the input was \( \frac{3}{2\pi} \) and the amplitude of the output was about 0.693. The graph of amplitude of the output (long term) as a function of the input frequency is referred to as the response curve of the system.

**Example 6.4.4** Consider an RLC circuit, with resistance 2 Ohms, capacitance \( \frac{1}{5} \) Farads, and inductance 1 Henry (this is the same circuit as in Example 6.4.3). Find the (long term) output amplitude as a function of the parameter \( \omega \) if the input is given by the forcing function \( f(t) = 5 \cos(\omega t) \), and graph this function. Also, determine the value of \( \omega \) that has the largest output amplitude. (Note: \( \omega \) is related to the frequency \( f \) by the formula \( f = \frac{\omega}{2\pi} \) as discussed in Section 3.2; we choose to work with \( \omega \) instead of \( f \) since the algebra is a bit simpler).

The differential equation is \( y''(t) + 2y'(t) + 5y(t) = 5 \cos(\omega t) \). Taking the Laplace transform of both sides we get
\[ s^2Y + 2sY + 5Y = 5\frac{s}{s^2 + \omega^2} \]
where we have assumed \( y(0) = 0 \) and \( y'(0) = 0 \). Solving for \( Y \) we get

\[
Y = \frac{1}{s^2 + 2s + 5} - \frac{s}{s^2 + \omega^2}
\]

with \( H(s) = \frac{1}{s^2 + 2s + 5} \) as the transfer function once again, and \( F(s) = \frac{5s}{s^2 + \omega^2} \) the input in the frequency domain. The solution in the time domain is

\[
y(t) = L^{-1} \left[ H(s)F(s) \right] = (h * f)(t) = \int_{u=0}^{t} h(t-u)f(u)du
\]

\[
= \int_{0}^{t} \frac{1}{2} e^{-(t-u)} \sin(2(t-u)) \cdot 5 \cos(\omega u)du
\]

\[
= \frac{(50 - 10\omega^2) \cos(\omega t) + 20\omega \sin(\omega t) - e^{-t}(25 + 5\omega^2) \sin(2t) + e^{-t}(10\omega^2 - 50) \cos(2t)}{50 - 12\omega^2 + 2\omega^4}
\]

Eliminating the transient terms (the ones with \( e^{-t} \)) and rearranging a bit we get

\[
y_{\text{long term}} = \frac{(50 - 10\omega^2)}{50 - 12\omega^2 + 2\omega^4} \cos(\omega t) + \frac{20\omega}{50 - 12\omega^2 + 2\omega^4} \sin(\omega t).
\]

The amplitude of \( y_{\text{long term}} \) would be

\[
\sqrt{\left( \frac{(50 - 10\omega^2)}{50 - 12\omega^2 + 2\omega^4} \right)^2 + \left( \frac{20\omega}{50 - 12\omega^2 + 2\omega^4} \right)^2} = \frac{5}{\sqrt{\omega^4 - 6\omega^2 + 25}}
\]

using the fact that the amplitude of \( C_1 \cos(\omega t) + C_2 \sin(\omega t) \) is \( \sqrt{C_1^2 + C_2^2} \) (Section 3.3) and a little computer algebra. Graphing the function \( f_{\omega}(\omega) = \frac{5}{\sqrt{\omega^4 - 6\omega^2 + 25}} \) we get the curve in Figure 6.7.
To find the maximum value of \( f_{\omega}(\omega) = \frac{5}{\sqrt{\omega^4 - 6\omega^2 + 25}} \), we find 
\[ f'_{\omega}(\omega) = -\frac{5}{2} \frac{4\omega^3 - 12\omega}{(\omega^4 - 6\omega^2 + 25)^{3/2}} \]
and then set \( f'_{\omega}(\omega) = 0 \) to get the three solutions \( \omega = 0 \), \( \omega = \sqrt{3} \), and \( \omega = -\sqrt{3} \). Thus, as can be seen from the graph of the response function, for positive \( \omega \) the maximum amplitude of the output occurs at \( \omega = \sqrt{3} \approx 1.7321 \). The frequency would be \( \frac{\sqrt{3}}{2\pi} \approx 0.27566 \) cycles per second.

We can interpret the response curve in terms of the concept of a filter. We say that values of \( \omega \) for which \( f_{\omega}(\omega) \) is large are passed through, and values of \( \omega \) for which \( f_{\omega}(\omega) \) is small are filtered out. Looking at Figure 6.7 from Example 6.4.4, we see that values of \( \omega \) larger than about \( 5 \) are effectively filtered out (corresponding to frequencies larger than \( \frac{5}{2\pi} \approx 0.79577 \) cycles per second).

### Poles and Zeroes

A pole of the transfer function \( H(s) \) is a value of \( s \) that makes the denominator of \( H(s) \) zero. A zero of \( H(s) \) is a value of \( s \) that makes the numerator of \( H(s) \) zero. The transfer function \( H(s) = \frac{1}{s^2 + 2s + 5} \) from Examples 6.4.3 and 6.4.4 has no zeroes (since the numerator can never be zero), and two poles at \( s = -1 + 2i \) and \( s = -1 - 2i \). Complex poles and zeroes need to be considered; this is why the word “plane” is used when we refer to the frequency domain as the \( s \)-plane.
Simple RLC circuits will always have either 2 poles or 1 pole (called a double pole) and no zeroes. More complex circuits, however, can have any number of poles and zeroes. The poles and zeroes are directly related to the response curve $f_{\omega}(\omega)$. Poles tend to boost certain frequencies, and zeroes tend to eliminate certain frequencies. Circuit designers use pole-zero plots to help them design circuits with a specific response curve.

In a pole-zero plot the poles are plotted in the complex plane using an “x” and the zeroes with an “o”. The pole-zero plot for the transfer function $H(s) = \frac{1}{s^2 + 2s + 5}$ is shown in Figure 6.8.

![Figure 6.8: Pole-Zero plot for $H(s) = \frac{1}{s^2 + 2s + 5}$](image)

In order to correctly interpret pole-zero plots, we need the following Theorem.

**Theorem 6.9** Consider a simple RLC circuit, with input frequency $\frac{\omega}{2\pi}$ (the forcing function has the form $f(t) = C_1 \cos(\omega t) + C_2 \sin(\omega t)$). If the poles of the transfer function $H(s)$ are at $a \pm bi$, with $a < 0$, then the frequency response curve is given by $f_{\omega}(\omega) = \frac{C}{\sqrt{(b+\omega)^2 + a^2} \sqrt{(b-\omega)^2 + a^2}}$ where $C$ is a positive constant, and its maximum occurs at $\omega = -\sqrt{b^2 - a^2}$ as long as $b > |a|$. If $b \leq |a|$, then the maximum occurs at $\omega = 0$.

Proof. The denominator of $H(s)$ can be factored as $(s - (a + bi))(s - (a - bi)) = s^2 - 2as +$
\(a^2 + b^2\). Therefore we can write \(H(s) = K \frac{1}{s^2 - 2as + a^2 + b^2}\), where \(K\) is some positive constant.

Using \(f(t) = C_1 \cos(\omega t) + C_2 \sin(\omega t)\) as the input, and hence \(F(s) = C_1 \frac{s}{s^2 + \omega^2} + C_2 \frac{\omega}{s^2 + \omega^2}\), we have \(Y(s) = K \frac{1}{s^2 - 2as + a^2 + b^2} \left( C_1 \frac{s}{s^2 + \omega^2} + C_2 \frac{\omega}{s^2 + \omega^2} \right)\). From this it follows that the amplitude of \(y_{\text{long term}}(t)\) is given by \(f_{\text{res}}(\omega) = \frac{K\sqrt{C_1^2 + C_2^2}}{\sqrt{(b+c)^2 + a^2} \sqrt{(b-c)^2 + a^2}}\) (to fill in the missing steps, take the inverse Laplace transform of \(Y(s)\), eliminate the transient terms, and find the magnitude of what is left as in the previous example). Setting the derivative of this expression equal to zero and solving for \(\omega\) gives \(\omega = \pm \sqrt{b^2 - a^2}\), and \(\omega = 0\). When \(b > |a|\), there are local maxima at \(\omega = \pm \sqrt{b^2 - a^2}\). When \(b \leq |a|\) there is a local maximum at \(\omega = 0\).

Note: If we want to graph the frequency response in terms of the frequency \(f = \frac{\omega}{2\pi}\) just make the substitution \(\omega = 2\pi f\) into \(f_{\omega}(\omega)\) to get \(f_{\text{freq}}(f) = \frac{C}{\sqrt{(b+2\pi f)^2 + a^2} \sqrt{(b-2\pi f)^2 + a^2}}\).

Notice that as \(a\) gets close to 0 in Theorem 6.9, then the maximum of the response occurs at a value very close to \(b\). When \(a = 0\), the transfer function \(H(s)\) gets infinitely large at \(\omega = b\); this corresponds to a circuit with no resistance, which is driven at its natural frequency. This is the condition of resonance. In real-world stable circuits, \(a\) will always be negative. Thus to design a filter which filters out all frequencies, except those near some specific value of \(\omega\), say \(\omega_0\), we put a pole at about \(a + \omega_0 i\), with \(a\) negative and small in absolute value compared to \(\omega_0\). Note: the exact maximum of the response curve of such a circuit will be a little bit less than \(\omega_0\).

**Example 6.4.5** Design an RLC circuit which allows frequencies near 1000 cycles per second to pass through, and filters out the other frequencies. Sketch the response curve.

1000 cycles per second corresponds to \(\omega = 2\pi(1000) \approx 6283\). We simply create a pole-zero plot with poles near the imaginary axis, and with imaginary component \(b\) near 6283. We choose \(a = -100\) so that it is small in absolute value compared to \(b\). See Figure 6.9. We know from the previous theorem that the response curve will have the form \(f_{\omega}(\omega) = \frac{C}{\sqrt{(b+\omega)^2 + a^2} \sqrt{(b-\omega)^2 + a^2}}\). Choosing \(C = 1\) for simplicity, and substituting \(\omega = 2\pi f\) (\(f\) represents frequency) into this expression we get \(f_{\text{freq}}(f) = \frac{1}{\sqrt{(b+2\pi f)^2 + a^2} \sqrt{(b-2\pi f)^2 + a^2}}\), which we graph in Figure 6.10. The denominator of the transfer function would be

\[
(s - (-100 + 6283i))(s - (-100 - 6283i)) = s^2 + 200s + 39486089.
\]
Thus \( H(s) = \frac{1}{s^2 + 200s + 39486089} \) and the corresponding left-side of the differential equation would be \( y'' + 200y' + 39486089 \). We could use an RLC circuit with \( L = 1 \), \( R = 200 \), and \( C = \frac{1}{39486089} \). Note: If we want the peak of the response function \( f_{\omega}(\omega) \) to be exactly at \( \omega = 6283 \) we would choose \( b = \sqrt{6283^2 + (-100)^2} \approx 6283.8 \).

**Figure 6.9:** Pole-zero plot to let \( \omega \) values near 6283 (frequencies near 1000) pass through

**Figure 6.10:** Response curve designed to pass only frequencies near 1000 cycles per second

**Comment:** We have focused on transfer functions with complex poles so far. Real-valued poles are also possible: they are plotted on the real axis in a pole-zero plot. Real-valued
poles tend to allow very low frequencies to pass through (since the complex component is zero).

**Comment:** Transfer functions for simple RLC circuits as described in this section do not have zeroes, but more complex circuits can have any number of zeroes (and poles). Zeroes tend to eliminate frequencies that are close to the complex component of the zero. In short, use poles to boost frequencies and zeroes to eliminate frequencies.

**Exercises** In problems 1-4 find the inverse Laplace transform of each function of \( s \). Do each problem two ways; once by expanding the expression first (partial fractions or computer algebra), and a second time using convolution.

1. \( \frac{1}{s-1} \frac{1}{s-2} \)
2. \( \frac{s}{s^2+1} \frac{1}{s+3} \)
3. \( \frac{1}{s^2} \frac{1}{s^2+4} \)
4. \( \frac{5}{(s-1)^2+4} \frac{2}{s+7} \)

For each RLC circuit in problems 5-8, use convolution to find an integral representation of the output \( i(t) \). Then evaluate the integral (using computer algebra if necessary) and eliminate the transient terms to obtain the long-term output and the amplitude of the long-term output.

5. \( L = 1, R = 4, C = \frac{1}{29} \), voltage source \( v_s(t) = 3 \sin(5t) \).
6. \( L = 1, R = 3, C = \frac{1}{2} \), voltage source \( v_s(t) = 3 \sin(5t) \).
7. \( L = 1, R = 3, C = \frac{1}{2} \), voltage source \( v_s(t) = 3 \sin(\omega t) \).
8. \( L = 1, R = 4, C = \frac{1}{29} \), voltage source \( v_s(t) = 3 \sin(\omega t) \).

Find the long-term output \( i(t) \) in the time domain, and find and sketch the response curve \( f_{\omega}(\omega) \), given the transfer function \( H(s) \) and input \( F(s) \) in the frequency domain. Also give the corresponding differential equation.

9. \( H(s) = \frac{1}{s^2+2s+10}, F(s) = \frac{s}{s^2+\omega^2} \).
10. \( H(s) = \frac{1}{s^2+4s+13}, F(s) = \frac{\omega}{s^2+\omega^2} \).

Use a pole-zero plot to design an RLC circuit which filters out all frequencies except those near the given frequency \( f \). Then design a second filter which has a wider pass-band (the response curve is wider), centered at roughly the same frequency \( f \) as the first filter. Give the values of \( R, L, \) and \( C \) for each case.
11. Frequency $f = 15000$ cycles per second.

12. Frequency $f = 0$ cycles per second.

13. Evaluate the convolution integral $\int_{u=0}^{u=t} \frac{1}{2} \sin(2u) \frac{1}{2} \sin(2(t-u))du$ without using computer algebra. You will need to use the trig identities $\sin(A \pm B) = \sin(A) \cos(B) \pm \cos(A) \sin(B)$ and $\cos(A \pm B) = \cos(A) \cos(B) \mp \sin(A) \sin(B)$. 