

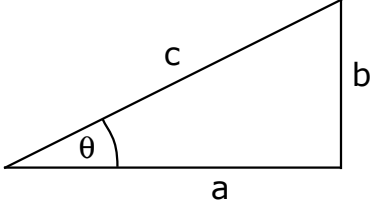
# Trig in Four Quadrants

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This document should lead you from the basic definitions of sine, cosine, and tangent to understanding their values in all four quadrants.

Start with basic definitions.

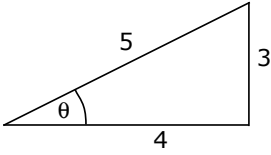
The basic trig triangle	Basic Definition	Definition using the diagram
	$\cos \theta = \frac{adj}{hyp}$ $\sin \theta = \frac{opp}{hyp}$ $\tan \theta = \frac{opp}{adj}$	$\cos \theta = \frac{a}{c}$ $\sin \theta = \frac{b}{c}$ $\tan \theta = \frac{b}{a}$

### Nomenclature:

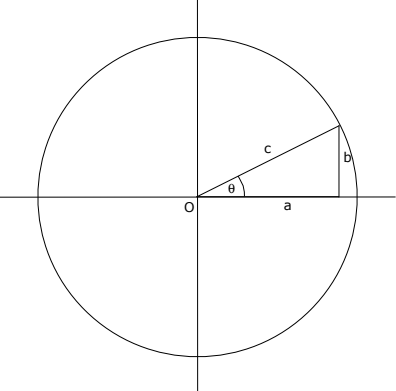
*adj* means the triangle leg adjacent to the angle  $\theta$ .  
*opp* means the triangle leg opposite to the angle  $\theta$ .  
*hyp* means the hypotenuse of the triangle.

$\cos \theta$  means cosine of the angle  $\theta$ .  
 $\sin \theta$  means sine of the angle  $\theta$ .  
 $\tan \theta$  means tangent of the angle  $\theta$ .

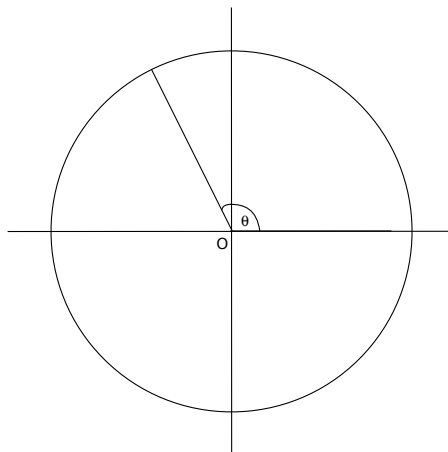
Let's take a 3-4-5 triangle.

The basic trig triangle	Basic Definition	Definition using the diagram
	$\cos \theta = \frac{adj}{hyp}$ $\sin \theta = \frac{opp}{hyp}$ $\tan \theta = \frac{opp}{adj}$	$\cos \theta = \frac{4}{5} = 0.8$ $\sin \theta = \frac{3}{5} = 0.6$ $\tan \theta = \frac{3}{4} = 0.75$

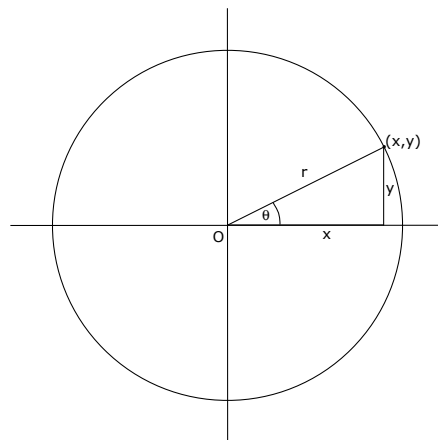
Now take the same general triangle and put a circle around it.

<p>The origin of the circle is at the lower left hand corner of the triangle.</p> <p>The radius of the circle is the length of the hypotenuse.</p> <p>The origin is symbolized by the letter O.</p>	
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Now what do we do if I change the angle.



To figure out what to do, we need to change the notation. We redraw the original triangle in a circle using the nomenclature of the axes. Note that the length of the bottom leg of the triangle in the circle,  $a$ , is the position on the  $x$  axis. The length of the side leg,  $b$ , is the position on the  $y$  axis. The radius,  $r$ , of the circle is the length of the hypotenuse.



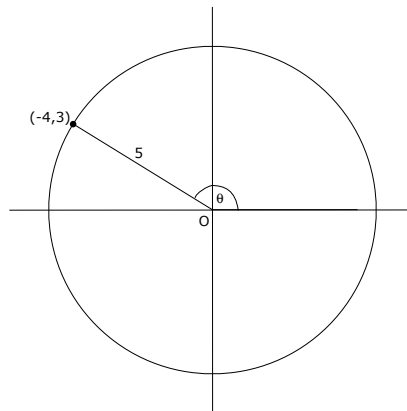
We now have the definitions in terms of the coordinates.

$$\cos \theta = \frac{x}{r}$$

$$\sin \theta = \frac{y}{r}$$

$$\tan \theta = \frac{y}{x}$$

From here we rethink what these formulas mean. Instead of thinking of  $x$  and  $y$  as the length of the legs, think of them as the coordinates of the point on the circle,  $(x,y)$  as indicated in the above diagram. Above we looked at the triangle whose point on the circle was  $(x,y) = (4,3)$ . Now let's look at the point on the circle  $(x,y) = (-4,3)$ .



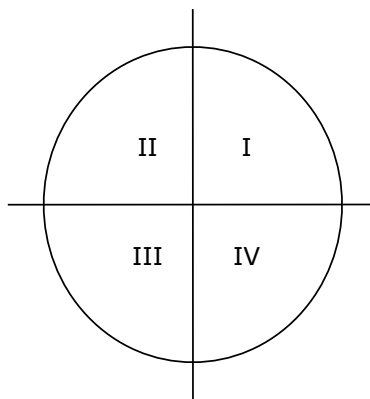
To figure out the values of its trig functions, use the more general formulas which use coordinates, not lengths.

$$\cos \theta = \frac{-4}{5} = -0.8$$

$$\sin \theta = \frac{3}{5} = 0.6$$

$$\tan \theta = \frac{3}{-4} = -0.75$$

So, by becoming a bit more abstract, i.e. not glued to the picture of the triangle to determine the trig functions of given angles, we have been able to find the values of those functions in quadrant II. The quadrants are labeled as



To expand our knowledge, let's look at the signs of the coordinates in the four quadrants.  $x$  values to the left of the  $y$  axis are negative; those to the right are positive.  $y$  values below the

$x$  axis are negative; those above are positive. Armed with that information, we build a sign table.

### Signs of coordinates

Quadrant	$x$	$y$
I	+	+
II	-	+
III	-	-
IV	+	-

Since the radius,  $r$ , is positive, the sine and cosine values follow the same pattern. We also add the signs of the tangents to the table since  $\tan \theta = \frac{y}{x} = \frac{\sin \theta}{\cos \theta}$ .

### Signs of trig functions

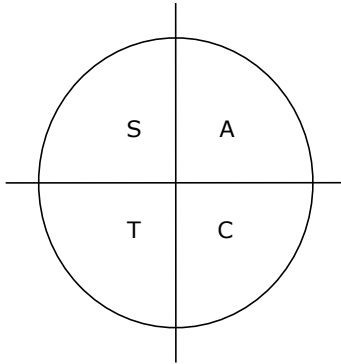
Quadrant	$\cos \theta$	$\sin \theta$	$\tan \theta$
I	+	+	+
II	-	+	-
III	-	-	+
IV	+	-	-

Note that the related trig functions follow the same pattern since they are just reciprocals of the above functions.

### Signs of reciprocal trig functions

Quadrant	$\sec \theta = \frac{1}{\cos \theta}$	$\csc \theta = \frac{1}{\sin \theta}$	$\cot \theta = \frac{1}{\tan \theta}$
I	+	+	+
II	-	+	-
III	-	-	+
IV	+	-	-

Now we construct a way to remember these signs. In the following circle, the  $A$  in quadrant I means that *All* signs are positive in quadrant I, the  $S$  means that only the **S**ine is positive in quadrant II, the  $T$  means that only the **T**angent is positive in quadrant III, and the  $C$  means that only the **C**osine is positive in quadrant IV.



Sometimes this diagram is referred to as “*All Students Take Calculus*” which is not true but makes it easy to remember which trig functions are positive in which quadrants. The order of the letters is counterclockwise around the circle starting in the first quadrant.

### Inverse Trig Functions

Now that we have looked at how to determine the signs of the trig functions given coordinates, how do we find the angles? Let’s return to our 3-4-5 triangle example. Recall  $(x,y) = (4,3)$  so

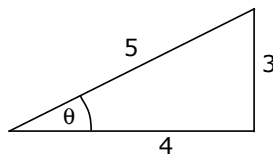
$$\cos \theta = \frac{4}{5} = 0.8 \qquad \sin \theta = \frac{3}{5} = 0.6 \qquad \tan \theta = \frac{3}{4} = 0.75$$

Now one questions is “What is the angle whose sine is 0.6?” Or, to put it another way, “What is the angle whose sine is 0.6?”

Let’s put it in the calculator (I use a TI-89) to find out the answer. I find that

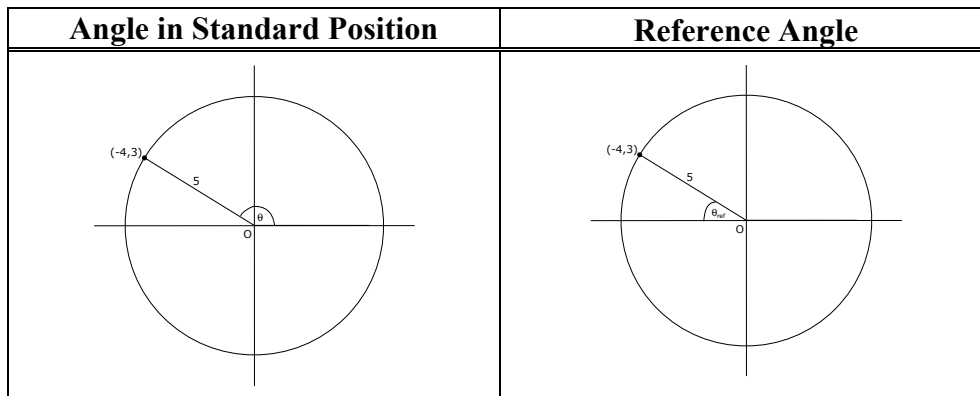
$$\theta = \sin^{-1} 0.6 = 36.87^\circ$$

That seems about right given the picture.



However, some students will get  $\theta = \sin^{-1} 0.6 = 0.6435$ . What happened? The first student’s calculator was in degree mode and the second student’s calculator was in radian mode. Just by looking at the diagram, we are expecting an answer somewhere near  $30^\circ \rightarrow 45^\circ$ . An answer of  $\theta = 0.6435^\circ$  is clearly wrong. That’s less than one degree which is just barely off the  $x$  axis. That’s how you know you are in the wrong mode. The answer makes no sense based on what your experience tells you.

Now let's look problem with the angle in the second quadrant. We will use the angle associated with  $(x,y) = (-4,3)$  and find the inverse cosine,  $\theta = \cos^{-1}(-0.8)$ . Note that I had to put the angle in parentheses to avoid any confusion. The question we are now asking is "What is the angle whose cosine is  $-0.8$ ?" . The answer is  $143.13$ . Huh? What happened to  $36.87^\circ$ . The answer is that the calculator knows that the angle is in the second quadrant. It is also giving the answer as the *angle in standard position*, i.e. the angle that starts at the  $x$  axis and continues counterclockwise to the line. The *reference angle* is the positive angle between the line and the  $x$  axis.



In our case the *angle in standard position* is  $143.13^\circ$  and its *reference angle* is  $36.87^\circ$ .

Finally, lets use the inverse tangent to see what that gives.

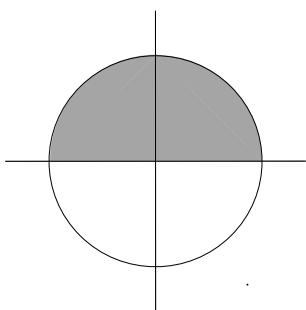
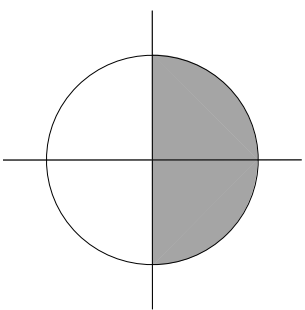
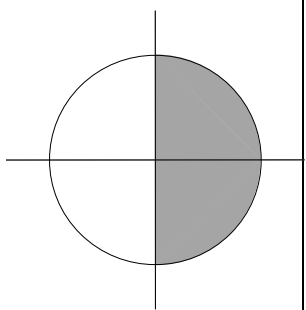
$$\theta = \tan^{-1}(-0.75) = -36.87^\circ$$

Now what happened? The answer we got is in quadrant IV but our angle is in quadrant II. Recall from the above chart that the tangent of an angle is negative in both quadrants II and IV. So there are really two answers to the question "What is the angle whose tangent is  $-0.75$ ?" The inverse tangent "function" is not actually a function. It has two correct answers. A function has only one. If we have a function  $y$  that is a function of  $x$ , i.e.  $y = f(x)$ , then if you tell me  $x$ , there is only one answer  $y$ . The calculator needs to use functions so it gives back only one of the two correct answers. We, not the calculator, know which quadrant the answer should be in and can get to the quadrant II answer by using a little geometry.

Let's first look at the inverse cosine, sine, and tangent. The angles returned by the TI-89 fall in the following ranges.

$$0^\circ \leq \theta = \cos^{-1} u < 180^\circ \quad -90^\circ \leq \theta = \sin^{-1} u < 90^\circ \quad -90^\circ \leq \theta = \tan^{-1} u < 90^\circ$$

These regions are shown in the shaded regions below.

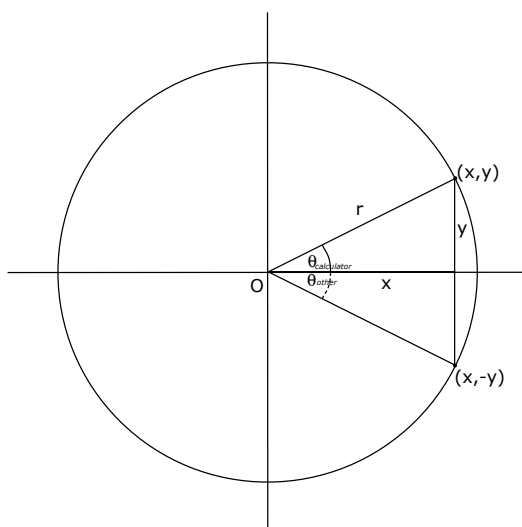
$\theta = \cos^{-1} u$	$\theta = \sin^{-1} u$	$\theta = \tan^{-1} u$
		

So, how do we figure out the second angle,  $\theta_{other}$ , that satisfies the equation?

Some “rules” for finding the second angle that satisfies a trig equation ( $\cos \theta = u$ ,  $\sin \theta = u$ , or  $\tan \theta = u$ ) and knowing the calculator angle,  $\theta_{calculator}$ , along with the signs of the  $x$  and  $y$  coordinates follow.

1)  $\theta = \cos^{-1} u$  :

$$\text{if } y < 0 \text{ then } \theta_{other} = -\theta_{calculator} \quad \text{since} \quad \cos \theta_{other} = \cos \theta_{calculator} = \frac{x}{r}$$



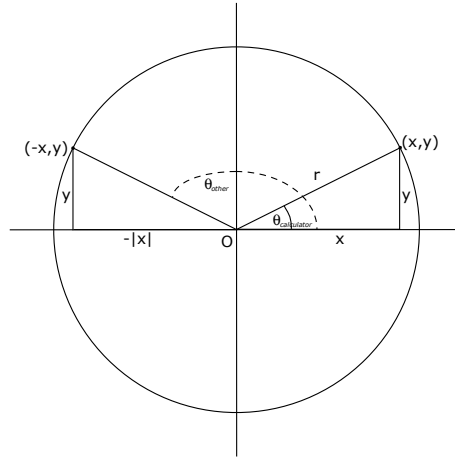
Note that to make  $0 \leq \theta_{other} < 360^\circ$ , add  $360^\circ$  to  $\theta_{other}$  :

$$\theta_{other} = 360^\circ - \theta_{calculator}$$

2)  $\theta = \sin^{-1} u$  :

if  $x < 0$  then  $\theta_{\text{other}} = 180^\circ - \theta_{\text{calculator}}$  since  $\sin \theta_{\text{other}} = \sin \theta_{\text{calculator}} = \frac{y}{r}$

Both angles have the same reference angle.



3)  $\theta = \tan^{-1} u$  :

if  $x < 0$  then  $\theta_{\text{other}} = 180^\circ + \theta_{\text{calculator}}$  since

$$\tan \theta_{\text{other}} = \frac{-y}{-x} = \tan \theta_{\text{calculator}} = \frac{y}{x}$$

