MTH 352 **The Laplace Transform (CH. 21, 22, 24) - Rev 2** **Fall 2009**

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Definition:

$$F(s)=\mathcal{L}\{f(t)\}=\int_0^\infty f(t)e^{-st}\,dt$$

There will be conditions on s such that the integral converges.

Recall $\int cf(x)dx = c\int f(x)dx$ hence $\mathcal{L}\{cf(t)\}=c\mathcal{L}\{f(t)\}$

Note that since the Laplace Transform is an integral, we know that

$$\mathcal{L}\{f(t)+g(t)\}=\mathcal{L}\{f(t)\}+\mathcal{L}\{g(t)\}$$

Also note that since the Laplace Transform is an integral

$$\mathcal{L}\{f(t)\bullet g(t)\}\neq\mathcal{L}\{f(t)\}\bullet\mathcal{L}\{g(t)\}$$

The integral of a product is ***NOT*** the product of the integrals.

Let's start building the table.

1) $f(t)=c$ where c is a constant and we noted above that $\mathcal{L}\{cf(t)\}=c\mathcal{L}\{f(t)\}$.

Therefore

$$F(s)=\int_0^\infty ce^{-st}\,dt=c\int_0^\infty e^{-st}\,dt=c\left(\frac{e^{-st}}{-s}\right)\Bigg|_{t=0}^{\infty}=-\frac{c}{s}\left\{\lim_{t\to\infty}e^{-st}-e^0\right\}$$

we know that $e^0=1$ but we need to be careful at the upper limit. If s>0 then $\lim_{t\to\infty}e^{-st}=0$. However, if s<0 then $\lim_{t\to\infty}e^{-st}=\infty$. Hence

$$\mathcal{L}\{c\}=\frac{c}{s}\qquad s>0$$

2) $f(t)=e^{at}$ where a is a constant.

$$F(s)=\int_0^\infty e^{at}e^{-st}\,dt=\int_0^\infty e^{-(s-a)t}\,dt=\left(\frac{e^{-(s-a)t}}{-(s-a)}\right)\Bigg|_{t=0}^{\infty}=-\frac{1}{(s-a)}\left\{\lim_{t\to\infty}e^{-(s-a)t}-e^0\right\}$$

This form is very similar to the one for a constant. If $s>a$ then $\lim_{t\to\infty}e^{-(s-a)t}=0$. However, if $s<a$ then $\lim_{t\to\infty}e^{-(s-a)t}=\infty$. Hence

$$\mathcal{L}\{e^{at}\}=\frac{1}{s-a}\qquad s>a\quad\text{(Note the minus sign in the denominator)}$$

3) $f(t)=\cos(bt)$ where b is a constant.
Let's recall Euler's theorem

$$e^{i\theta}=\cos\theta+i\sin\theta \qquad \text{equivalently} \qquad e^{-i\theta}=\cos\theta-i\sin\theta$$

If we solve for $\cos\theta$ and $\sin\theta$, we get

$$\cos\theta=\frac{e^{i\theta}+e^{-i\theta}}{2} \qquad \text{and} \qquad \sin\theta=\frac{e^{i\theta}-e^{-i\theta}}{2i}$$

We can use these relationships to find the Laplace Transforms of $\cos\theta$ and $\sin\theta$.

$$\mathcal{L}\{\cos bt\}=\int_0^\infty \frac{e^{ibt}+e^{-ibt}}{2}e^{-st}\,dt=\frac{1}{2}\left\{\int_0^\infty e^{ibt}e^{-st}\,dt+\int_0^\infty e^{-ibt}e^{-st}\,dt\right\}$$

But we just found $\mathcal{L}\{e^{at}\}=\frac{1}{s-a}$. We can therefore write down that

$$\mathcal{L}\{\cos bt\}=\int_0^\infty \frac{e^{ibt}+e^{-ibt}}{2}e^{-st}\,dt=\frac{1}{2}\left\{\frac{1}{s-ib}+\frac{1}{s-ib}\right\}$$

We combine the terms using the LCD of $LCD=(s+ib)(s-ib)=s^2+b^2$.

Therefore

$$\mathcal{L}\{\cos bt\}=\frac{1}{2}\left\{\frac{s+ib}{s^2+b^2}+\frac{s-ib}{s^2+b^2}\right\}=\frac{1}{2}\left\{\frac{2s}{s^2+b^2}\right\}=\frac{s}{s^2+b^2}$$

Using the same technique, you prove that

$$\mathcal{L}\{\sin bt\}=\frac{b}{s^2+b^2}$$

4) The hyperbolic sine and cosine behave similarly.

$$\cosh x=\frac{e^x+e^{-x}}{2} \qquad \text{and} \qquad \sinh x=\frac{e^x-e^{-x}}{2}$$

$$\mathcal{L}\{\cosh bt\}=\frac{s}{s^2-b^2} \qquad \text{and} \qquad \mathcal{L}\{\sinh bt\}=\frac{b}{s^2-b^2}$$

5) One more that you can prove using integration by parts is

$$\mathcal{L}\{t^n\}=\frac{n!}{s^{n+1}} \qquad \text{where } n!=n\cdot(n-1)\cdot(n-2)\ldots3\cdot2\cdot1$$

Find the Laplace Transforms of the following functions of time.

$f(t)=5t+2$	$f(t)=10t^2-5$
$f(t)=e^{3t}$	$f(t)=e^{5t-7}$
$f(t)=\cos(3t)$	$f(t)=\sin(5t+2)$
$f(t)=\cosh(3t)$	$f(t)=\sinh(5t+2)$

Note: $\sin(A+B)=\sin A\cos B+\cos A\sin B$ $\qquad$ $\sinh(A+B)=\sinh A\cosh B+\cosh A\sinh B$

$\cos(A+B)=\cos A\cos B-\sin A\sin B$ $\qquad$ $\cosh(A+B)=\cosh A\cosh B+\sinh A\sinh B$

Check out Trig Expand in the F2 menu of the TI-89.