Phasors From a Differential Equations Point of View

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Original Differential Equation (2nd order, linear, homogeneous) for a series RLC circuit with voltage source $v(t)$:

$$v(t) = v_L(t) + v_C(t) + v_R(t) \qquad (1)$$

$$LQ'' + RQ' + \frac{1}{C}Q = v(t) \qquad (2a)$$

This equation is also written as

$$Li' + Ri + \frac{1}{C}\int i(t)\,dt = v(t) \qquad (2b)$$

Assume the voltage is

$$v(t) = V_m \sin \omega t \qquad (3)$$

The resulting current is assumed to be out of phase with the voltage so is written*

$$i(t) = I_m \sin(\omega t - \phi) \qquad (4)$$

To avoid the mess resulting from using trig functions, recall

Euler's identity

$$e^{j\theta} = \cos\theta + j\sin\theta \qquad (5)$$

So

$$\sin(\theta) = \operatorname{Im}\left(e^{j\theta}\right) \qquad (6)$$

The voltage and current are then rewritten as

$$v(t) = V_m e^{j\omega t} \qquad (7)$$

$$i(t) = I_m e^{j(\omega t - \phi)} \qquad (8)$$

Substitute equations (7) and (8) into equation (2b):

$$V_m = \left(j\omega L + R + \frac{1}{j\omega C} \right) I_m e^{-j\phi} \qquad (9)$$

Define the impedance, Z:

$$Z = e^{-j\phi}\left(j\omega L + R + \frac{1}{j\omega C} \right) \qquad (10)$$

So that

$$V_m = Z I_m \qquad (11)$$

Since V_m and I_m are the maximum values of the voltage and current, respectively, they are real hence Z must be real.   First, we define the inductive and capacitive reactances so the formulas are easier to handle:

$$X_L = \omega L \qquad X_C = \frac{1}{\omega C} \qquad (12a,b)$$

This statement allows us to find the phase angle, ϕ: $$Z = e^{-j\phi}\left(R + j\left(X_L - X_C\right)\right) \quad (13)$$

The magnitude of Z is then $$Z = \sqrt{R^2 + \left(X_L - X_C\right)^2} \quad (14a)$$

The phase, ϕ, is found from $R + j\left(X_L - X_C\right) = e^{j\phi} = \cos\phi + j\sin\phi$ which gives

$$\tan\phi = \frac{X_L - X_C}{R} \quad (14b)$$

Now let's look at the individual element voltages:

$$Z = e^{-j\phi}\left(jX_L + R - jX_C\right) = e^{-j\phi}\left(e^{j\pi/2}X_L + R + e^{-j\pi/2}X_C\right)$$

So, in exponential form we have

$$v_L = I_m e^{j(\omega t-\phi)}\left(jX_L\right) = I_m e^{j(\omega t-\phi)}\left(e^{j\pi/2}X_L\right) = e^{j(\omega t-\phi+\pi/2)}I_m X_L \quad (15a)$$

$$v_C = I_m e^{j(\omega t-\phi)}\left(-jX_C\right) = e^{j(\omega t-\phi-\pi/2)}I_m X_C \quad (15b)$$

$$v_R = e^{j(\omega t-\phi)}I_m R \quad (15c)$$

To return to reality, we take the *Im* of each of the above three equations.

$$v_L = \mathrm{Im}\left(e^{j(\omega t-\phi+\pi/2)}I_m X_L\right) = I_m X_L \sin\left(\omega t - \phi + 90^0\right) \quad (16a)$$

$$v_C = \mathrm{Im}\left(e^{j(\omega t-\phi-\pi/2)}I_m X_C\right) = I_m X_C \sin\left(\omega t - \phi - 90^0\right) \quad (16b)$$

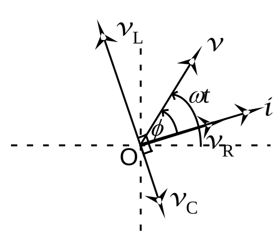
$$v_R = \mathrm{Im}\left(e^{j(\omega t-\phi)}I_m R\right) = I_m R\sin\left(\omega t - \phi\right) \quad (16c)$$

And recall that $$v = V_m \sin\left(\omega t\right) \quad (16d)$$

In other words the voltage drop across the inductor leads the current by 90º ($\pi/2$) and the capacitor lags by 90º ($\pi/2$). The current and voltage drop across the resistor are in phase.

Note that all four equations (16) have *sines* in them. If you think of each of these equations as representing the *y* component of a vector, then we can construct a vector diagram. This diagram is called a *phasor* diagram. Note that, due to the ωt, the entire phasor diagram rotates around the origin counterclockwise with frequency $f = \frac{\omega}{2\pi}$. Also note that we can write the voltage law in vector form.

$$\vec{v} = \vec{v}_L + \vec{v}_C + \vec{v}_R$$



*Cathay and Nasar, Basic Electrical Engineering, Schaum's Outline Series, McGraw-Hill, 1984, p54.