

**Nonhomogeneous Equations – Variation of Parameters**  
**Prof. Townsend**                      **Fall 2010**

Derrick and Grossman, Elementary Differential Equations 4th ed., Section 3.5

Assume the particular solution is written

$$x_p(t) = u_1(t)x_1(t) + u_2(t)x_2(t) \tag{1}$$

It is the solution of the equation

$$\ddot{x}_p + a_1\dot{x}_p + a_0x_p = F(t) \tag{2}$$

At this point I will suppress the explicit  $t$  dependence to reduce notational clutter. The dot notation means derivative with respect to time.

We need both first and second derivative so equation (1) to plug into equation (2) to determine equations for  $u_1(t)$  and  $x_2(t)$ . We use the product rule several times.

$$\frac{d(uv)}{dt} = u \frac{dv}{dt} + v \frac{du}{dt} \tag{3}$$

$$\dot{x}_p = \dot{u}_1x_1 + u_1\dot{x}_1 + \dot{u}_2x_2 + u_2\dot{x}_2 \tag{4}$$

Derrick and Grossman show that (p. 139 problem 29) we can set

$$\dot{u}_1x_1 + \dot{u}_2x_2 = 0 \tag{5}$$

They show that setting  $\dot{u}_1x_1 + \dot{u}_2x_2 = z(t)$  does not change the overall result.

Then equation (4) becomes

$$\dot{x}_p = u_1\dot{x}_1 + u_2\dot{x}_2 \tag{6}$$

The second derivative is found to be

$$\ddot{x}_p = \dot{u}_1\dot{x}_1 + u_1\ddot{x}_1 + \dot{u}_2\dot{x}_2 + u_2\ddot{x}_2 + \dot{u}_2\dot{x}_2 \tag{7}$$

Plug equations (6) and (7) into equation (2).

$$\dot{u}_1\dot{x}_1 + u_1\ddot{x}_1 + \dot{u}_2\dot{x}_2 + u_2\ddot{x}_2 + a_1(u_1\dot{x}_1 + u_2\dot{x}_2) + a_0(u_1x_1 + u_2x_2) = F(t) \tag{8}$$

Gather coefficients of derivatives of  $u_1(t)$  and  $u_2(t)$ .

$$\dot{u}_1\dot{x}_1 + \dot{u}_2\dot{x}_2 + u_1(\ddot{x}_1 + a_1\dot{x}_1 + a_0x_1) + u_2(\ddot{x}_2 + a_1\dot{x}_2 + a_0x_2) = F(t) \tag{9}$$

The last two terms on the left hand side vanish because  $x_1(t)$  and  $x_2(t)$  are solutions of the homogeneous equations  $\ddot{x}_1 + a_1\dot{x}_1 + a_0x_1 = 0$  and  $\ddot{x}_2 + a_1\dot{x}_2 + a_0x_2 = 0$ .

Therefore

$$\dot{x}_1 \dot{u}_1 + \dot{x}_2 \dot{u}_2 = F(t) \quad (10)$$

Gathering our two equations for  $\dot{u}_1$  and  $\dot{u}_2$  (equations 5 and 10) we have

$$\dot{u}_1 x_1 + \dot{u}_2 x_2 = 0 \quad (5)$$

$$\dot{x}_1 \dot{u}_1 + \dot{x}_2 \dot{u}_2 = F(t) \quad (10)$$

We have two equations in two unknowns. Rewrite them as a matrix equation.

$$\begin{pmatrix} x_1 & x_2 \\ \dot{x}_1 & \dot{x}_2 \end{pmatrix} \begin{pmatrix} \dot{u}_1 \\ \dot{u}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ F(t) \end{pmatrix} \quad (11)$$

The solution is

$$\begin{pmatrix} \dot{u}_1 \\ \dot{u}_2 \end{pmatrix} = \frac{1}{W(t)} \begin{pmatrix} x_2 & -x_2 \\ -\dot{x}_1 & \dot{x}_1 \end{pmatrix} \begin{pmatrix} 0 \\ F(t) \end{pmatrix} \quad (12)$$

where  $W(t)$  is called the Wronskian and is the determinant of the matrix in equation (11).

$$W(t) = x_1 \dot{x}_2 - x_2 \dot{x}_1 \quad (13)$$

The derivatives of the coefficient functions,  $\dot{u}_1$  and  $\dot{u}_2$ , are found from multiplying out equation (12).

$$\dot{u}_1 = \frac{1}{W(t)} (-x_2) F(t) = -\frac{x_2 F(t)}{W(t)} \quad (14a)$$

$$\dot{u}_2 = \frac{1}{W(t)} (\dot{x}_1) F(t) = \frac{x_1 F(t)}{W(t)} \quad (14b)$$

**Another way to find the full solution of the differential equation** (problem 30 p 140).

Note that D&G consider the case of non-constant coefficients. Assume we know one solution of the homogeneous equation,  $x_1(t)$ .

Set 
$$x(t) = u(t)x_1(t) \quad (15)$$

Define 
$$h(t) = e^{\int a_1 dt} = e^{a_1 t} \quad (16)$$

Then 
$$u(t) = C_1 + C_2 \int \frac{h}{x_1^2} dt + \int \frac{h}{x_1^2} \left[ \int \frac{x_1 F}{h} \right] dt \quad (17)$$

Note that if the original equation is homogeneous, then  $F(t) = 0$  and the second solution of the

homogeneous equation is 
$$u(t) = C_1 + C_2 \int \frac{h}{x_1^2} dt \quad (18)$$