Newton's Law of Cooling

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Our text writes that Newton's Law of Cooling obeys the differential equation

$$\frac{dT}{dt} = -k\left(T - T_m\right)$$

where T is the time-dependent temperature of the object, T_m is the temperature of the medium which contains the object, and k is a materials dependent constant.

This equation can be solved in many ways.

1) First, we look at the equation as a separable equation. Change the variable from just T to $T - T_m$. Then the equation can be written as

$$\frac{d\left(T - T_m\right)}{dt} = -k\left(T - T_m\right)$$

This form leads to

$$\frac{d\left(T - T_m\right)}{\left(T - T_m\right)} = -k dt$$

The solution is

$$T - T_m = Ce^{-kt}$$

i.e.

$$T = T_m + Ce^{-kt}$$

2) Now look at the equation as a linear equation.

$$\frac{dT}{dt} + kT == kT_m$$

Using the formula for the solution of a linear equation gives

$$I\left(t\right) = e^{kt} \qquad \text{Integrating factor}$$

$$T\left(t\right) = \frac{1}{e^{kt}}\left\{C + \int e^{kt} kT_m \, dt\right\}$$

Since T_m and k are constants, they pull out of the integral giving

$$T\left(t\right) = \frac{1}{e^{kt}}\left\{C + kT_m \int e^{kt} \, dt\right\} = \frac{1}{e^{kt}}\left\{C + kT_m \frac{e^{kt}}{k}\right\}$$

A little algebra gives

$$T\left(t\right) = \frac{1}{e^{kt}}\left\{C + T_m e^{kt}\right\} = T_m + \frac{C}{e^{kt}} = T_m + Ce^{-kt}$$

The general solution is then given by

$$T = T_m + Ce^{-kt}$$

Next we give the constants C and k physical meaning.

3) We apply the initial condition $T(0) == T_0$ so

$$T_0 = T_m + C$$

Hence

$$C = T_0 - T_m$$

The solution is therefore

$$T(t) = T_m + (T_0 - T_m)e^{-kt}$$

Note that when *t=0*, the solution gives

$$T(t) = T_m + (T_0 - T_m) = T_0$$

and for long times,

$$T(t) \to T_m$$

as one would expect.

How long is a long time? EETs say that e^{-5} is close enough to 0, so when $t = \frac{k}{5}$, the long time condition is reached. The constant k can now be given physical meaning. It is the inverse of the time constant, τ, for the problem.

$$k = \frac{1}{\tau}$$

A physically meaningful way of writing the solution of Newton's Law of Cooling is therefore

$$T(t) = T_m + (T_0 - T_m)e^{-\frac{t}{\tau}}$$

Every symbol has a physical meaning.

t	The time
$T(t)$	The temperature of the object as a function of time.
T_m	The temperature of the medium.
T_0	The initial temperature of the object.
τ	The time constant of the object in the medium. When $t > 5\tau$, the temperature of the object has essentially reached the temperature of the medium.