

Harmonic Identities

Derrick and Grossman, Elementary Differential Equations 4th ed. P 130

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MTH 352 Fall 2005

Identity #1: $a \cos \omega t + b \sin \omega t = A \cos(\omega t - \delta)$ (1)

With $A = \sqrt{a^2 + b^2}$ $\cos \delta = \frac{a}{A}$ $\sin \delta = \frac{b}{A}$ $\tan \delta = \frac{b}{a}$

When determining δ by inverse trig functions, you know which quadrant is the correct quadrant since you know the sign of both the sine and the cosine. The inverse *sin* and *tan* return angles from -90° to $+90^\circ$ and the inverse *cos* from 0° to 180° . To get an angle in Quadrant III take the inverse *tan* then add or subtract 180° . Be careful of radians vs. degrees as ωt is usually in radians.

Quadrant	Angles	$\cos \delta = \frac{a}{A}$	$\sin \delta = \frac{b}{A}$
I	0° to 90°	+	+
II	90° to 180°	-	+
III	180° to 270°	-	-
IV	270° to 360°	+	-

Proof:

Use $\cos(\omega t - \delta) = \cos \omega t \cos \delta + \sin \omega t \sin \delta$
 So $A \cos(\omega t - \delta) = \cos \omega t (A \cos \delta) + \sin \omega t (A \sin \delta)$
 Compare with $A \cos(\omega t - \delta) = a \cos \omega t + b \sin \omega t$

Identity #2: $a \cos \omega t + b \sin \omega t = A \sin(\omega t + \phi)$ (2)

With $A = \sqrt{a^2 + b^2}$ $\cos \phi = \frac{b}{A}$ $\sin \phi = \frac{a}{A}$ $\tan \phi = \frac{a}{b}$

When determining δ by inverse trig functions, you know which quadrant is the correct quadrant since you know the sign of both the sine and the cosine. The inverse *sin* and *tan* return angles from -90° to $+90^\circ$ and the inverse *cos* from 0° to 180° . To get an angle in Quadrant III take the inverse *tan* then add or subtract 180° . Be careful of radians vs. degrees as ωt is usually in radians.

Quadrant	Angles	$\cos \phi = \frac{b}{A}$	$\sin \phi = \frac{a}{A}$
I	0° to 90°	+	+
II	90° to 180°	-	+
III	180° to 270°	-	-
IV	270° to 360°	+	-

Proof:

Use $\sin(\omega t + \phi) = \cos \omega t \sin \phi + \sin \omega t \cos \phi$
 So $A \sin(\omega t + \phi) = \cos \omega t (A \sin \phi) + \sin \omega t (A \cos \phi)$
 Compare with $A \sin(\omega t + \phi) = a \cos \omega t + b \sin \omega t$