

Harmonic Identities

Derrick and Grossman, Elementary Differential Equations 4th ed. P 130
by Prof. Townsend MTH 352 Fall 2010

Identity #1:

$$a \cos \omega t + b \sin \omega t = A \cos(\omega t - \delta)$$

With $A = \sqrt{a^2 + b^2}$ $\cos \delta = \frac{a}{A}$ $\sin \delta = \frac{b}{A}$ $\tan \delta = \frac{b}{a}$

When determining δ by inverse trig functions, you know which quadrant is the correct quadrant since you know the sign of both the sine and the cosine. The inverse *sin* and *tan* return angles from -90° to $+90^\circ$ and the inverse *cos* from 0° to 180° . To get an angle in Quadrant II or III take the inverse *tan* then add or subtract 180° . Be careful of radians vs. degrees as ωt is usually in radians.

Quadrant	Angles	$\cos \delta = \frac{a}{A}$	$\sin \delta = \frac{b}{A}$
I	0° to 90°	+	+
II	90° to 180°	-	+
III	180° to 270°	-	-
IV	270° to 360°	+	-

Proof:

Use	$\cos(\omega t - \delta) = \cos \omega t \cos \delta + \sin \omega t \sin \delta$
Rearrange	$A \cos(\omega t - \delta) = (A \cos \delta) \cos \omega t + (A \sin \delta) \sin \omega t$
Compare with	$A \cos(\omega t - \delta) = (a) \cos \omega t + (b) \sin \omega t$

Identity #2:

$$a \cos \omega t + b \sin \omega t = A \sin(\omega t + \theta)$$

With $A = \sqrt{a^2 + b^2}$ $\cos \theta = \frac{b}{A}$ $\sin \theta = \frac{a}{A}$ $\tan \theta = \frac{a}{b}$

When determining θ by inverse trig functions, you know which quadrant is the correct quadrant since you know the sign of both the sine and the cosine. The inverse *sin* and *tan* return angles from -90° to $+90^\circ$ and the inverse *cos* from 0° to 180° . To get an angle in Quadrant II or III take the inverse *tan* then add or subtract 180° . Be careful of radians vs. degrees as ωt is usually in radians.

Quadrant	Angles	$\cos \theta = \frac{b}{A}$	$\sin \theta = \frac{a}{A}$
I	0° to 90°	+	+
II	90° to 180°	-	+
III	180° to 270°	-	-
IV	270° to 360°	+	-

Proof: Use $\sin(\omega t + \theta) = \cos \omega t \sin \theta + \sin \omega t \cos \theta$
 Rearrange $A \sin(\omega t + \theta) = (A \sin \theta) \cos \omega t + (A \cos \theta) \sin \omega t$
 Compare with $A \sin(\omega t + \theta) = (a) \cos \omega t + (b) \sin \omega t$

Note: Look in the F2 menu of the TI-89. You will find a trig section at the end. *tCollect* and *tExpand* are the commands that accomplish the above. Note that *tCollect* gives you back inverse tangents. To copy just that part do the following

- 1) Put the equation in the command line. Example: $\cos(2x - \tan^{-1}(3/5))$.
- 2) With the arrow cursor, move between the last two right parentheses.
- 3) Holding down the up arrow button between 2nd and ESC. Arrow left to just in front of the $\tan^{-1}(3/5)$. The $\tan^{-1}(3/5)$ should now be highlighted.
- 4) Copy it using F1/5. The entire line is now highlighted.
- 5) Paste it over the current command line using F1/6.
- 6) Press diamond-enter to evaluate $\tan^{-1}(3/5)$. Make sure you are in the desired trig mode.