

## Exam Scaling

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I look at two particular numbers to scale from one set of grades to another. Those numbers are the average value, also called the *mean*, and the *standard deviation* which is a measure of the spread in the data. Refer to your MTH 112/122 book for details of elementary statistics.

Let the grades be represented by  $x_i$  where the letter  $i$  stands for student number.

The average grade for  $N$  students is then

$$\bar{x} = \frac{1}{N} \sum_{i=1}^N x_i$$

where the bar over the  $x$  means *average value*. The symbol  $\sum_{i=1}^N x_i$  means add up all  $N$  values for  $x_i$ .

The standard deviation,  $\sigma$ , is given by

$$\sigma^2 = \frac{1}{N-1} \sum_{i=1}^{N-1} (x_i - \bar{x})^2$$

Note that in what follows it does not matter whether  $N-1$  or just  $N$  is used. The derivation follows for using either a sample or the full population.

Let me scale your grades for exam 2,  $x_i$ , such that they have the same  $\bar{x}$  and  $\sigma$  as exam 1.

Your new grade,  $y_m$ , is written as

$$y_m = ax_{2_m} + b$$

Let  $\bar{x}_1$  and  $\sigma_1$  be those for exam 1. There were  $N_1$  exam scores. I want exam 2 with  $N_2$  exam scores to have the same statistics as those for exam 1. In other words, set

$$\bar{y} = \bar{x}_1 \quad \text{and} \quad \sigma_y = \sigma_1$$

These two equivalencies provide two equations for two unknowns,  $a$  and  $b$ . The resulting scaled grade  $y_m$  is

$$y_m = \frac{\sigma_1}{\sigma_2} (x_{2_m} - \bar{x}_2) + \bar{x}_1$$

## Derivation

	Exam 1	Exam 2	Scaled Exam 2
<b>Average Value</b>	$\bar{x}_1 = \frac{1}{N_1} \sum_{i=1}^{N_1} x_{1_i}$	$\bar{x}_2 = \frac{1}{N_2} \sum_{k=1}^{N_2} x_{2_k}$	$\bar{y} = \frac{1}{N_2} \sum_{m=1}^{N_2} y_m$
<b>Standard Deviation</b>	$\sigma_1^2 = \frac{1}{N_1 - 1} \sum_{i=1}^{N_1-1} (x_{1_i} - \bar{x}_1)^2$	$\sigma_2^2 = \frac{1}{N_2 - 1} \sum_{k=1}^{N_2-1} (x_{2_k} - \bar{x}_2)^2$	$\sigma_y^2 = \frac{1}{N_2 - 1} \sum_{m=1}^{N_2-1} (y_m - \bar{y})^2$

**Problem:**

Make exam 2 have the same statistics as exam 1

$$\bar{y} = \bar{x}_1 \quad \text{and} \quad \sigma_y = \sigma_1 \quad \text{where} \quad y_m = ax_{2_m} + b$$

**Solution:**

**Average value:**

Plug  $y_m = ax_{2_m} + b$  into  $\bar{y} = \frac{1}{N_2} \sum_{m=1}^{N_2} y_m$

$$\bar{y} = \frac{1}{N_2} \sum_{m=1}^{N_2} (ax_{2_m} + b) = a \left( \frac{1}{N_2} \sum_{m=1}^{N_2} x_{2_m} \right) + b$$

$$\bar{y} = a\bar{x}_2 + b$$

**Standard deviation:**

Plug in  $y_m = ax_{2_m} + b$  into  $\sigma_y^2 = \frac{1}{N_2 - 1} \sum_{m=1}^{N_2-1} (y_m - \bar{y})^2$

$$\sigma_y^2 = \frac{1}{N_2 - 1} \sum_{m=1}^{N_2-1} (ax_{2_m} + b - \bar{y})^2$$

Add and subtract  $a\bar{x}_2$  inside the summation to make the calculation easier.

$$\sigma_y^2 = \frac{1}{N_2 - 1} \sum_{m=1}^{N_2-1} \left( a \{ x_{2_m} - \bar{x}_2 \} - \{ \bar{y} - a\bar{x}_2 - b \} \right)^2$$

Recall from above that  $\bar{y} = a\bar{x}_2 + b$ . Therefore the second term vanishes and we have

$$\sigma_y^2 = \frac{1}{N_2 - 1} \sum_{m=1}^{N_2-1} \left( a \{ x_{2_m} - \bar{x}_2 \} \right)^2 = a^2 \left\{ \frac{1}{N_2 - 1} \sum_{m=1}^{N_2-1} (x_{2_m} - \bar{x}_2)^2 \right\}$$

$$\sigma_y^2 = a^2 \sigma_2^2$$

Taking the square root of both sides we have

$$\sigma_y = a\sigma_2$$

**Collect equations and solve for  $a$  and  $b$  :**

$$\bar{y} = a\bar{x}_2 + b \quad \text{and} \quad \sigma_y = a\sigma_2$$

Now set  $\bar{y} = \bar{x}_1$  and  $\sigma_y = \sigma_1$  .

We therefore have

$$\bar{x}_1 = a\bar{x}_2 + b \quad \text{and} \quad \sigma_1 = a\sigma_2$$

Solve for  $a$  and  $b$  .

$$a = \frac{\sigma_1}{\sigma_2} \quad \text{and} \quad b = \bar{x}_1 - a\bar{x}_2 = \bar{x}_1 - \frac{\sigma_1}{\sigma_2}\bar{x}_2$$

The equation to be used for  $y_m = ax_{2_m} + b$  is

$$y_m = \left( \frac{\sigma_1}{\sigma_2} \right) x_{2_m} + \left( \bar{x}_1 - \frac{\sigma_1}{\sigma_2} \bar{x}_2 \right)$$

**Final result:**

$$y_m = \frac{\sigma_1}{\sigma_2} (x_{2_m} - \bar{x}_2) + \bar{x}_1$$

Q.E.D.