

M344 - ADVANCED ENGINEERING MATHEMATICS

Lecture 9: Orthogonal Functions and Trigonometric Fourier Series

Before learning to solve partial differential equations, it is necessary to know how to approximate arbitrary functions by infinite series, using special families of functions. This process is called Fourier approximation, and the families of functions we use are called **orthogonal families**.

Orthogonal Families of Functions

Def 1 An infinite collection of functions $S = \{\Phi_0(t), \Phi_1(t), \Phi_2(t), \dots\} \equiv \{\Phi_n(t)\}_{n=0}^{\infty}$ is said to be an **orthogonal set** on the interval $[\alpha, \beta]$ if

$$\int_{\alpha}^{\beta} \Phi_m(t)\Phi_n(t)dt = 0, \text{ whenever } m \neq n. \quad (1)$$

Suppose $S = \{\Phi_n(t)\}_{n=0}^{\infty}$ is an orthogonal set on $[\alpha, \beta]$, and we want to represent an arbitrary function $f(t)$ by a series of the form

$$f(t) = \sum_{n=0}^{\infty} a_n \Phi_n(t) = a_0 \Phi_0(t) + a_1 \Phi_1(t) + \dots + a_n \Phi_n(t) + \dots \quad (2)$$

If we assume that this series converges to $f(t)$ on $\alpha \leq t \leq \beta$, it is then easy to find the coefficients a_m . For any $m = 0, 1, 2, \dots$, using equations (1) and (2),

$$\begin{aligned} \int_{\alpha}^{\beta} f(t)\Phi_m(t)dt &= \int_{\alpha}^{\beta} \left(\sum_{n=0}^{\infty} a_n \Phi_n(t) \right) \Phi_m(t)dt = \sum_{n=0}^{\infty} a_n \int_{\alpha}^{\beta} \Phi_n(t)\Phi_m(t)dt \\ &= a_m \int_{\alpha}^{\beta} \Phi_m^2(t)dt; \end{aligned}$$

and therefore, a formula for the m th coefficient in the Fourier Series for $f(t)$ is

$$a_m = \frac{\int_{\alpha}^{\beta} f(t)\Phi_m(t)dt}{\int_{\alpha}^{\beta} \Phi_m^2(t)dt.}$$

We are going to need to find Fourier approximations for arbitrary functions $f(t)$ in terms of different sets of orthogonal functions. In computing the coefficients by integration, life is sometimes simplified if we know that the function f satisfies certain properties.

Review of Even and Odd Functions

Def 2 A function $f(t)$ is called an **even function** if $f(-t) = f(t)$ for all t in the domain of f ; it is called an **odd function** if $f(-t) = -f(t)$ for all t in the domain of f .

If f and g are both odd functions, or both even functions, then the product $f(t)g(t)$ is even; and if one is even and the other is odd then the product is odd.

Even functions are symmetric about the y -axis. Some examples of even functions are constants, any even-degree polynomial like t^2 or $t^4 + 3t^8$, and $\cos(\omega t)$. For any even function $f(t)$, $\int_{-L}^L f(t)dt = 2 \int_0^L f(t)dt$. Odd functions are symmetric about the origin. Examples are odd-degree polynomials t , $2t^3 - 5t^{11}$, and $\sin(\omega t)$. If $g(t)$ is an odd function, $\int_{-L}^L g(t)dt = 0$.

The two figures below, showing graphs of an even function $f(x) = -0.35x^4 + x^2 + 1$ and an odd function $g(x) = \frac{x^3 - 3x}{2}$, illustrate the above properties.

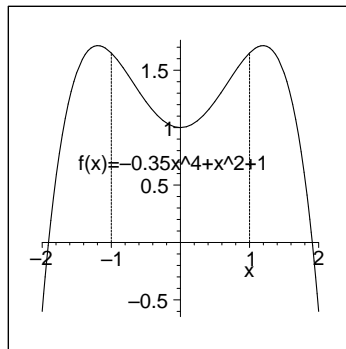


Figure 1: Even function f

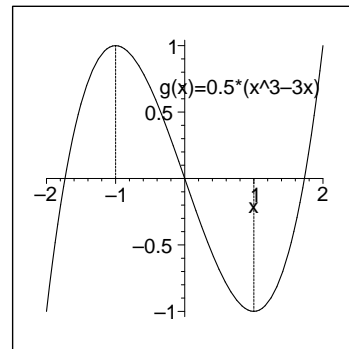


Figure 2: Odd function g

Two functions, which you may not have seen before, should be added to this list. They are the hyperbolic sine, $\sinh(t)$, and hyperbolic cosine, $\cosh(t)$. We will be using these functions when we solve partial differential equations. They are defined as follows:

$$\sinh(t) = \frac{e^t - e^{-t}}{2}, \quad \cosh(t) = \frac{e^t + e^{-t}}{2}.$$

Check that they satisfy the differentiation formulas

$$\frac{d}{dt} \sinh(t) = \cosh(t), \quad \frac{d}{dt} \cosh(t) = \sinh(t).$$

One other property of these hyperbolic functions, that will be needed later, is that $\cosh(t)$ is never equal to 0, and $\sinh(t) = 0$ only if $t = 0$.

Example 1 Show that $\sinh(t)$ is an odd function and $\cosh(t)$ is even.

To show that a function f is odd, we need to evaluate it at $-t$ and show that this is the same as the value of $-f$ at t :

$$\sinh(-t) = \frac{e^{(-t)} - e^{-(-t)}}{2} = \frac{e^{-t} - e^t}{2} = -\frac{e^t - e^{-t}}{2} = -\sinh(t).$$

To show that $\cosh(t)$ is even, we show

$$\cosh(-t) = \frac{e^{(-t)} + e^{-(-t)}}{2} = \frac{e^{-t} + e^t}{2} = \cosh(t).$$

Example 2 Show that if f is an even function and g is odd, then the product $f \cdot g$ is an odd function.

Again we need to show that the product function $f \cdot g$, evaluated at $-t$, is equal to the negative of its value at t .

$$f \cdot g(-t) \equiv f(-t) \cdot g(-t) = f(t) \cdot -g(t) \equiv -f \cdot g(t).$$

Trigonometric Fourier Series

We are now ready to study the approximation of functions in terms of the particular orthogonal set

$$\mathcal{S} = \left\{ 1, \cos\left(\frac{\pi t}{L}\right), \sin\left(\frac{\pi t}{L}\right), \cos\left(\frac{2\pi t}{L}\right), \sin\left(\frac{2\pi t}{L}\right), \dots, \cos\left(\frac{n\pi t}{L}\right), \sin\left(\frac{n\pi t}{L}\right), \dots \right\}.$$

The series for f , in terms of this set, is usually written in the form

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi t}{L}\right) + b_n \sin\left(\frac{n\pi t}{L}\right). \quad (3)$$

It is called the **Trigonometric Fourier Series** for $f(t)$ on the interval $[-L, L]$. Note that each function in the set \mathcal{S} has period $2L$; that is $\Phi_n(t + 2L) = \Phi_n(t)$ for all t ; therefore, if $f(t)$ is represented by its Trigonometric Fourier Series (3), it will be a periodic function with period $2L$.

Orthogonality of the set \mathcal{S}

To show that the set $\mathcal{S} = \{\Phi_n(t)\}_{n=0}^{\infty} \equiv \{1, \cos(\frac{\pi t}{L}), \sin(\frac{\pi t}{L}), \cos(\frac{2\pi t}{L}), \dots\}$ is an orthogonal set on $[-L, L]$, we need to show that $\int_{-L}^L \Phi_n(t) \Phi_m(t) dt = 0$ whenever Φ_n and Φ_m are two different functions in \mathcal{S} .

Showing that $\int_{-L}^L \sin(\frac{n\pi t}{L}) \cdot 1 dt = 0$ and $\int_{-L}^L \sin(\frac{n\pi t}{L}) \cos(\frac{m\pi t}{L}) dt = 0$ for any integers m and n is easy since in both cases the integrand is an *odd* function; that is, it is the product of an even function times an odd function. Using the fact that $\cos(\frac{n\pi t}{L})$ is an even function, and $\sin(n\pi) = 0$ for any integer n ,

$$\int_{-L}^L \cos\left(\frac{n\pi t}{L}\right) \cdot 1 dt = 2 \int_0^L \cos\left(\frac{n\pi t}{L}\right) dt = \frac{2L}{n\pi} (\sin(n\pi) - \sin(0)) = 0.$$

Showing that the product of two sine functions or two cosine functions integrates to zero is done using trig substitutions and is left to the exercises.

In the next lecture we will have a theorem stating which functions $f(t)$ have convergent Fourier Series. For these functions, the coefficient of $\Phi_n(t)$ in the series is equal to $\frac{\int_{\alpha}^{\beta} f(t)\Phi_n(t)dt}{\int_{\alpha}^{\beta} \Phi_n^2(t)dt}$. For the trigonometric Fourier Series this implies that for $n \geq 1$,

$$a_n = \frac{\int_{-L}^L f(t) \cos(\frac{n\pi t}{L})dt}{\int_{-L}^L (\cos(\frac{n\pi t}{L}))^2 dt}, \quad b_n = \frac{\int_{-L}^L f(t) \sin(\frac{n\pi t}{L})dt}{\int_{-L}^L (\sin(\frac{n\pi t}{L}))^2 dt}.$$

You are going to show in the exercises that $\int_{-L}^L (\cos(\frac{n\pi t}{L}))^2 dt = \int_{-L}^L (\sin(\frac{n\pi t}{L}))^2 dt = L$ for any $n \geq 1$; therefore, the formulas for the coefficients a_n and b_n are

$$a_n = \frac{1}{L} \int_{-L}^L f(t) \cos(\frac{n\pi t}{L})dt, \quad b_n = \frac{1}{L} \int_{-L}^L f(t) \sin(\frac{n\pi t}{L})dt.$$

When $n = 0$, the coefficient of $\Phi_0(t) = 1$ in the Fourier Series is $\frac{\int_{-L}^L f(t)dt}{\int_{-L}^L 1^2 dt}$; and since $\int_{-L}^L 1 dt = 2L$, if we use the formula for a_n to compute the constant coefficient, it must be divided by 2. Note that the Fourier Series in equation (3) starts with the term $a_0/2$. It is also helpful to recognize that the constant $a_0/2$ must be the average value of the function $f(t)$ on the interval $[L, L]$. This is true since each of the sine and cosine functions has average value zero. (Remember from Calculus that the **average value** of a function $f(t)$ on the interval $a \leq t \leq b$ is defined as $\frac{1}{b-a} \int_a^b f(t)dt$.)

In the next lecture we will summarize properties of the Trigonometric Fourier Series, and state for which functions $f(t)$ these series converge; but first we can do a simple example.

Example 3 Find a trigonometric Fourier Series for the piecewise continuous function

$$f(t) = \begin{cases} -1 & \text{if } -L \leq t < 0 \\ 1 & \text{if } 0 \leq t \leq L \end{cases}$$

First, notice that f is an odd function, and therefore the coefficients $a_n = \frac{1}{L} \int_{-L}^L f(t) \cos(\frac{n\pi t}{L})dt$ are all zero since they are the integral of the product of an even function times an odd function. Note that the coefficients b_n are integrals of even functions, so that

$$\begin{aligned} b_n &= \frac{1}{L} \int_{-L}^L f(t) \sin(\frac{n\pi t}{L})dt = \frac{2}{L} \int_0^L f(t) \sin(\frac{n\pi t}{L})dt \\ &= \frac{2}{L} \int_0^L 1 \cdot \sin(\frac{n\pi t}{L})dt = \left(\frac{2}{L}\right)\left(\frac{L}{n\pi}\right)(-\cos(n\pi) + \cos(0)) = \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{4}{n\pi} & \text{if } n \text{ is odd} \end{cases} \end{aligned}$$

The Fourier Series for $f(t)$ can therefore be written in the form

$$f(t) \sim \sum_{n=1,3,\dots}^{\infty} \frac{4}{n\pi} \sin(\frac{n\pi t}{L}) = \frac{4}{\pi} \left(\frac{\sin(\frac{\pi t}{L})}{1} + \frac{\sin(\frac{3\pi t}{L})}{3} + \frac{\sin(\frac{5\pi t}{L})}{5} + \dots \right).$$

Choosing a particular value, $L = 1$, an approximation to $f(t)$ was obtained by taking terms in the sum out to $n = 21$. A graph of this finite approximation is shown in Figure (3). It shows clearly that the Fourier Series is a periodic function, of period $2L = 2$.

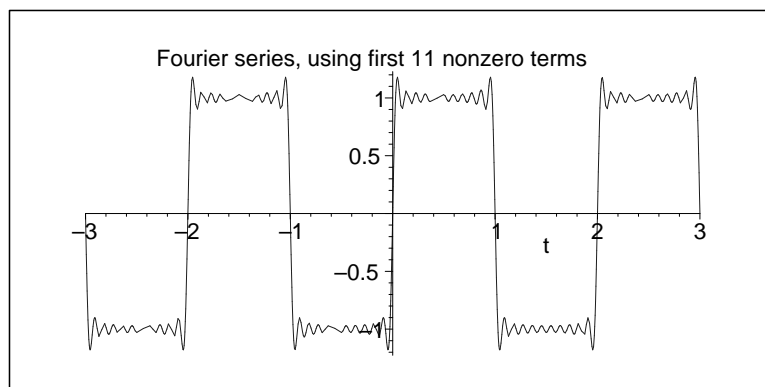


Figure 3:

Practice Problems:

- Determine whether each function below is even, odd, or neither.

(a) $1 + t^2$	Ans: even	(d) $1 + 2 \cosh(t)$	Ans: even
(b) $\sin(2t) + 6t$	Ans: odd	(e) $1 + t^3$	Ans: neither
(c) e^t	Ans: neither		
- Use the trig identity $\cos(A)\cos(B) = \frac{1}{2}(\cos(A-B) + \cos(A+B))$ to show that if $m \neq n$, then $\int_{-L}^L \cos\left(\frac{n\pi t}{L}\right)\cos\left(\frac{m\pi t}{L}\right)dt = 0$.
Hint: the integrand is an even function.
- Use the trig identity $\sin(A)\sin(B) = \frac{1}{2}(\cos(A-B) - \cos(A+B))$ to show that if $m \neq n$, then $\int_{-L}^L \sin\left(\frac{n\pi t}{L}\right)\sin\left(\frac{m\pi t}{L}\right)dt = 0$.
- Show that $\int_{-L}^L (\cos\left(\frac{n\pi t}{L}\right))^2 dt = \int_{-L}^L (\sin\left(\frac{n\pi t}{L}\right))^2 dt = L$ for any integer $n > 0$. Hint: use the trig formulas $\cos^2(x) = \frac{1+\cos(2x)}{2}$ and $\sin^2(x) = \frac{1-\cos(2x)}{2}$.
- * Let $f(t) = t^2$ for $-1 \leq t \leq 1$, and assume that $f(t)$ is *periodic* of period 2.
 - Draw a graph of this function on $-3 \leq t \leq 3$.
 - Find a formula for the coefficients of the Fourier Series for $f(t)$.
 - Using all terms out to $n = 3$, sketch a graph of the Fourier Series approximation to $f(t)$ on the interval $[-3, 3]$.
- * Approximating a function $f(t)$ by a Maclaurin series $\sum_0^\infty \frac{f^{(n)}(0)}{n!} t^n$ is equivalent to using the set of functions $\mathcal{S} = \{1, t, t^2, t^3, \dots\}$. Is this set of functions orthogonal on $[-1, 1]$? Either prove that it is, or find two integers $m \neq n$ such that $\int_{-1}^1 t^m \cdot t^n dt \neq 0$.
- * There is a set $\mathcal{S} = \{P_0(t), P_1(t), P_2(t), \dots\}$ of polynomials, called the **Legendre polynomials**, which forms an orthogonal set on the interval $[-1, 1]$. For each integer $n = 0, 1, 2, \dots$, $P_n(t)$ is a polynomial of degree n . Look up **Legendre polynomials** on the web, or in a text book, and find the formulas for $P_0(t), P_1(t), \dots, P_5(t)$. Also find a recursion formula that allows you to use $P_{n-2}(t)$ and $P_{n-1}(t)$ to determine $P_n(t)$.