Now that we know the general solution of the Bessel equation of order $\nu = 0$ is $x(t) = C_1 J_0(t) + C_2 Y_0(t)$, and have the ability to compute values of this function at any $t$ using MAPLE, the next step is to show that other differential equations can be converted into this form by a change of variables. An interesting example of this type is the Aging Spring equation which was solved by the ordinary power series method in Lecture 4.

**Conversion of the Aging Spring Equation to Bessel’s equation**

It will only be possible to do the conversion for the *undamped* aging spring equation:

$$mx'' + ke^{-\eta t}x = 0. \tag{1}$$

We will make a change of independent variable of the form

$$s = \alpha e^{\beta t} \tag{2}$$

which will convert equation (1) into the form of Bessel’s equation of order 0:

$$s^2X''(s) + sX'(s) + s^2X(s) = 0. \tag{3}$$

The constants $\alpha$ and $\beta$ will be chosen to make the conversion work. First note that by differentiating (2)

$$\frac{ds}{dt} = \beta(\alpha e^{\beta t}) = \beta s.$$

Also, the aging coefficient $e^{-\eta t}$ can be easily written in terms of $s$, by noting that $e^{-\eta t} = (e^{\beta t})^{-\eta/\beta} = \left(\frac{s}{\alpha}\right)^{-\eta/\beta}$.

If we let $X(s) \equiv x(t)$, then the derivatives $x'$ and $x''$ can be found in terms of the new variable $X$ using the Chain Rule and the product rule, as follows:

$$x' = \frac{dx}{dt} = \frac{d}{dt}X(s(t)) = \frac{dX}{ds} \cdot \frac{ds}{dt} = \beta sX'(s)$$

and

$$x'' = \frac{d}{dt}(\beta sX'(s)) = \beta s \frac{d}{dt}(X'(s)) + \beta X'(s) \frac{ds}{dt}$$

$$= \beta sX''(s) \cdot \frac{ds}{dt} + \beta X'(s) \beta s = (\beta s)^2 X''(s) + \beta^2 sX'(s).$$
Now equation (1) becomes

\[ mx'' + ke^{-ut} x = m \left( \beta^2 s^2 X''(s) + \beta^2 sX'(s) \right) + k \left( \frac{s}{\alpha} \right)^{-\frac{n}{\beta}} X(s) = 0. \]

Then dividing by \( m\beta^2 \), and choosing the value \( \beta = -\frac{n}{2} \) so that \(-\frac{n}{\beta} = 2\),

\[ mx'' + ke^{-ut} x = s^2 X''(s) + sX'(s) + \frac{k}{m\beta^2} \cdot \frac{s^2}{\alpha^2} X(s) = 0. \]

To make this into Bessel’s equation (3), we need \( \frac{k}{m\beta^2 \alpha^2} = 1 \), so choose the second constant \( \alpha \) so that \( \alpha^2 = \frac{k}{m\beta^2} \); that is, let \( \alpha = \sqrt{\frac{k}{m \beta}} = \sqrt{\frac{k}{m} \cdot \frac{2}{\eta}} \).

Then the equation becomes \( s^2 X'' + sX' + s^2 X = 0 \), and it has the solution \( X(s) = C_1 J_0(s) + C_2 Y_0(s) \). Substituting \( s = \alpha e^{\beta t} = \sqrt{\frac{k}{m}} \cdot \frac{2}{\eta} e^{-\frac{n}{2}t} \) back into this solution,

\[ x(t) \equiv X(s(t)) = C_1 J_0(\sqrt{\frac{k}{m}} \cdot \frac{2}{\eta} e^{-\frac{n}{2}t}) + C_2 Y_0(\sqrt{\frac{k}{m}} \cdot \frac{2}{\eta} e^{-\frac{n}{2}t}). \]  

(4)

This form of the solution gives much more information than did the series solution found in Example 2 in Lecture 4. This is illustrated by the following example.

**Example 1** Solve the initial value problem

\[ x'' + 4e^{-0.2t} x = 0, \quad x(0) = 1, \quad x'(0) = 0. \]  

(5)

This is the equation solved in Example 2 in Lecture 4, except that the damping coefficient has been set to zero. From equation (4), with the argument

\[ \sqrt{\frac{k}{m}} \cdot \frac{2}{\eta} e^{-\frac{n}{2}t} = \sqrt{\frac{4}{1}} \cdot \frac{2}{0.2} e^{-0.1t} = 20e^{-0.1t}, \]

we see that the general solution is

\[ x(t) = C_1 J_0(20e^{-0.1t}) + C_2 Y_0(20e^{-0.1t}). \]

Notice that as \( t \to \infty \), the argument \( 20e^{-0.1t} \to 0 \). We showed previously that \( Y_0(x) \to -\infty \) as \( x \to 0^+ \). This means that as \( t \to \infty \), \( Y_0(20e^{-0.1t}) \to -\infty \); therefore, \( x(t) \) will tend to \( \pm \infty \), depending only on the sign of the constant \( C_2 \). If \( C_2 = 0 \) the value of \( x(t) \) will tend to 0 as \( t \to \infty \). Furthermore, at time \( t = 0 \), the argument of the Bessel functions is equal to 20. Since \( x(t) \) is a sum
of Bessel functions varying over the reverse interval $20$ to $0$, the number of oscillations it makes will be approximately equal to the number of zero crossings of $J_0(t)$ on the interval from $0$ to $20$. Looking at the graph of $J_0$, it crosses the axis $5$ times in this $t$-interval.

The coefficients $C_1$ and $C_2$ can be found by using the initial conditions. First, the derivative of $x(t)$ must be computed. We know that $\frac{d}{dt}J_0(t) = -J_1(t)$ and it can also be shown that $\frac{d}{dt}Y_0(t) = -Y_1(t)$; therefore, by the Chain Rule, $\frac{d}{dt}J_0(s(t)) = -J_1(s(t)) \frac{ds}{dt}$. Similarly, $\frac{d}{dt}Y_0(s(t)) = -Y_1(s(t)) \frac{ds}{dt}$. Now we can write

$$x'(t) = \frac{d}{dt} \left( C_1 J_0(20e^{-0.1t}) + C_2 Y_0(20e^{-0.1t}) \right)$$

$$= -C_1 J_1(20e^{-0.1t})(-2e^{-0.1t}) - C_2 Y_1(20e^{-0.1t})(-2e^{-0.1t}).$$

The initial conditions give the two equations

$$x(0) = C_1 J_0(20) + C_2 Y_0(20) = 1.0, \quad x'(0) = 2C_1 J_1(20) + 2C_2 Y_1(20) = 0.$$ 

MAPLE can be used to obtain the four values $J_0(20), J_1(20), Y_0(20)$ and $Y_1(20)$ (see Problem #1 in the exercises for Lecture 7). Solving the two linear equations, we find $C_1 \approx 5.1997$, and $C_2 \approx 2.0996$. This already tells us that the mass will oscillate about its equilibrium position about $5$ times, and then tend to $-\infty$ as $t \to \infty$. Figure (1) shows a graph of the solution on the interval $0 < t < 15$.

![Graph of $x(t) = 5.1997J_0(20e^{-0.1t}) + 2.0996Y_0(20e^{-0.1t})$](image)

Figure 1: Graph of $x(t) = 5.1997J_0(20e^{-0.1t}) + 2.0996Y_0(20e^{-0.1t})$
If DEplot is used to draw a graph of a numerical approximation to the solution of the initial-value problem 5, it produces the graph shown below which is essentially identical to the graph in Figure 1.

Figure 2: Solution of the IVP in equation 5, using DEplot

Practice Problems:

1. Find a solution of the initial-value problem

   \[2x'' + 18e^{-0.5t}x = 0, \quad x(0) = 0, \quad x'(0) = 2\]

   in terms of Bessel functions.

2. Draw a graph of the solution found in #1.

3. Use the MAPLE commands

   ```maple
   with(DEtools);
   de1:= 2.0*diff(x(t),t$2) + 18.0*exp(-0.5*t)*x(t)=0;
   DEplot({de1},[x(t)],t=0..15,[[x(0)=0,D(x)(0)=2]],stepsize=0.05);
   ```

   to draw a numerical approximation to the solution of the IVP in Problem 1.