M344 - ADVANCED ENGINEERING MATHEMATICS

Lecture 8: Converting d.e.s to Bessel's Equation

Now that we know the general solution of the Bessel equation of order $\nu = 0$ is $x(t) = C_1 J_0(t) + C_2 Y_0(t)$, and have the ability to compute values of this function at any t using MAPLE, the next step is to show that other differential equations can be converted into this form by a change of variables. An interesting example of this type is the Aging Spring equation which was solved by the ordinary power series method in Lecture 4.

Conversion of the Aging Spring Equation to Bessel's equation

It will only be possible to do the conversion for the *undamped* aging spring equation:

$$mx'' + ke^{-\eta t}x = 0. \tag{1}$$

We will make a change of independent variable of the form

$$s = \alpha e^{\beta t} \tag{2}$$

which will convert equation (1) into the form of Bessel's equation of order 0:

$$s^2 X''(s) + sX'(s) + s^2 X(s) = 0. \tag{3}$$

The constants α and β will be chosen to make the conversion work. First note that by differentiating (2)

$$\frac{ds}{dt} = \beta(\alpha e^{\beta t}) = \beta s.$$

Also, the aging coefficient $e^{-\eta t}$ can be easily written in terms of s, by noting that $e^{-\eta t} = (e^{\beta t})^{(-\eta/\beta)} = \left(\frac{s}{\alpha}\right)^{-\frac{\eta}{\beta}}$.

If we let $X(s) \equiv x(t)$, then the derivatives x' and x'' can be found in terms of the new variable X using the Chain Rule and the product rule, as follows:

$$x' = \frac{dx}{dt} = \frac{d}{dt}X(s(t)) = \frac{dX}{ds} \cdot \frac{ds}{dt} = \beta s X'(s)$$

and

$$x'' = \frac{d}{dt}(\beta s X'(s)) = \beta s \frac{d}{dt}(X'(s)) + \beta X'(s)\frac{ds}{dt}$$

$$= \beta s X''(s) \cdot \frac{ds}{dt} + \beta X'(s)\beta s = (\beta s)^2 X''(s) + \beta^2 s X'(s).$$

Now equation (1) becomes

$$mx'' + ke^{-\eta t}x = m\left(\beta^2 s^2 X''(s) + \beta^2 s X'(s)\right) + k\left(\frac{s}{\alpha}\right)^{-\frac{\eta}{\beta}} X(s) = 0.$$

Then dividing by $m\beta^2$, and choosing the value $\boldsymbol{\beta = -\frac{\eta}{2}}$ so that $-\frac{\eta}{\beta} = 2$,

$$mx'' + ke^{-\eta t}x = s^2 X''(s) + sX'(s) + \frac{k}{m\beta^2} \cdot \frac{s^2}{\alpha^2} X(s) = 0.$$

To make this into Bessel's equation (3), we need $\frac{k}{m\beta^2\alpha^2} = 1$, so choose the second constant α so that $\alpha^2 = \frac{k}{m\beta^2}$; that is, let $\boldsymbol{\alpha = \frac{\sqrt{k/m}}{-\beta} = \sqrt{k/m} \cdot \frac{2}{\eta}}$. Then the equation becomes $s^2X'' + sX' + s^2X = 0$, and it has the solution $X(s) = C_1 J_0(s) + C_2 Y_0(s)$. Substituting $s = \alpha e^{\beta t} = \sqrt{k/m} \cdot \frac{2}{\eta} e^{-\frac{\eta}{2}t}$ back into this solution,

$$x(t) \equiv X(s(t)) = C_1 J_0(\sqrt{k/m} \cdot \frac{2}{\eta} e^{-\frac{\eta}{2}t}) + C_2 Y_0(\sqrt{k/m} \cdot \frac{2}{\eta} e^{-\frac{\eta}{2}t}). \tag{4}$$

This form of the solution gives much more information than did the series solution found in Example 2 in Lecture 4. This is illustrated by the following example.

Example 1 *Solve the initial value problem*

$$x'' + 4e^{-0.2t}x = 0, \quad x(0) = 1, \quad x'(0) = 0. \tag{5}$$

This is the equation solved in Example 2 in Lecture 4, except that the damping coefficient has been set to zero. From equation (4), with the argument

$$\sqrt{k/m} \cdot \frac{2}{\eta} e^{-\frac{\eta}{2}t} = \sqrt{4/1} \cdot \frac{2}{0.2} e^{-0.1t} = 20e^{-0.1t},$$

we see that the general solution is

$$x(t) = C_1 J_0(20e^{-0.1t}) + C_2 Y_0(20e^{-0.1t}).$$

Notice that as $t \to \infty$, the argument $20e^{-0.1t} \to 0$. We showed previously that $Y_0(x) \to -\infty$ as $x \to 0^+$. This means that as $t \to \infty$, $Y_0(20e^{-0.1t}) \to -\infty$; therefore, $x(t)$ will tend to $\pm\infty$, depending only on the sign of the constant C_2. If $C_2 = 0$ the value of $x(t)$ will tend to 0 as $t \to \infty$. Furthermore, at time $t = 0$, the argument of the Bessel functions is equal to 20. Since $x(t)$ is a sum *of Bessel functions varying over the reverse interval* 20 *to* 0*, the number of oscillations it makes will be approximately equal to the number of zero crossings of* $J_0(t)$ *on the interval from* 0 *to* 20*. Looking at the graph of* J_0*, it crosses the axis* 5 *times in this* t*-interval.*

The coefficients C_1 *and* C_2 *can be found by using the initial conditions. First, the derivative of* $x(t)$ *must be computed. We know that* $\frac{d}{dt}J_0(t) = -J_1(t)$ *and it can also be shown that* $\frac{d}{dt}Y_0(t) = -Y_1(t)$*; therefore, by the Chain Rule,* $\frac{d}{dt}J_0(s(t)) = -J_1(s(t))\frac{ds}{dt}$*. Similarly,* $\frac{d}{dt}Y_0(s(t)) = -Y_1(s(t))\frac{ds}{dt}$*. Now we can write*

$$x'(t) = \frac{d}{dt}\left(C_1J_0(20e^{-0.1t}) + C_2Y_0(20e^{-0.1t})\right)$$
$$= -C_1J_1(20e^{-0.1t})(-2e^{-0.1t}) - C_2Y_1(20e^{-0.1t})(-2e^{-0.1t}).$$

The initial conditions give the two equations

$$x(0) = C_1J_0(20) + C_2Y_0(20) = 1.0, \quad x'(0) = 2C_1J_1(20) + 2C_2Y_1(20) = 0.$$

MAPLE can be used to obtain the four values $J_0(20), J_1(20), Y_0(20)$ *and* $Y_1(20)$ *(see Problem # 1 in the exercises for Lecture 7). Solving the two linear equations, we find* $C_1 \approx 5.1997$*, and* $C_2 \approx 2.0996$*. This already tells us that the mass will oscillate about its equilibrium position about* 5 *times, and then tend to* $-\infty$ *as* $t \to \infty$*. Figure (1) shows a graph of the solution on the interval* $0 < t < 15$*.*

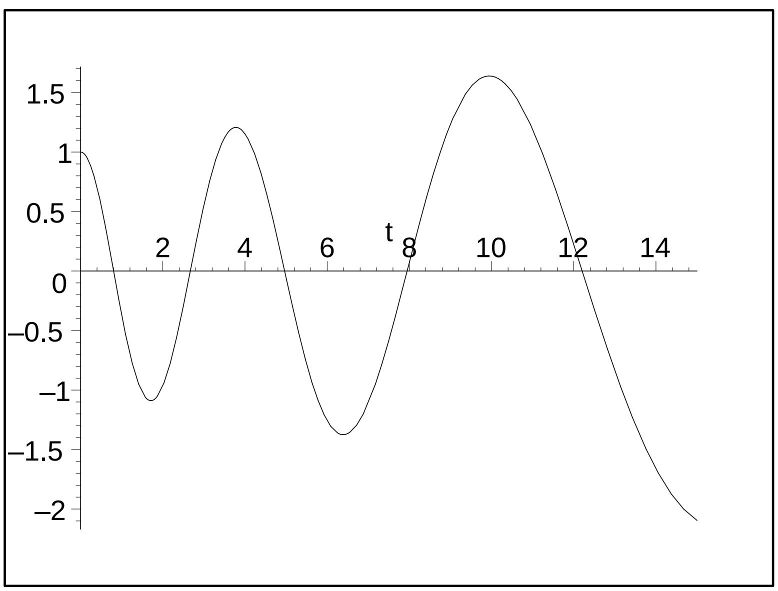

Figure 1: Graph of $x(t) = 5.1997J_0(20e^{-0.1t}) + 2.0996Y_0(20e^{-0.1t})$

If `DEplot` *is used to draw a graph of a numerical approximation to the solution of the initial-value problem 5, it produces the graph shown below which is essentially identical to the graph in Figure 1.*

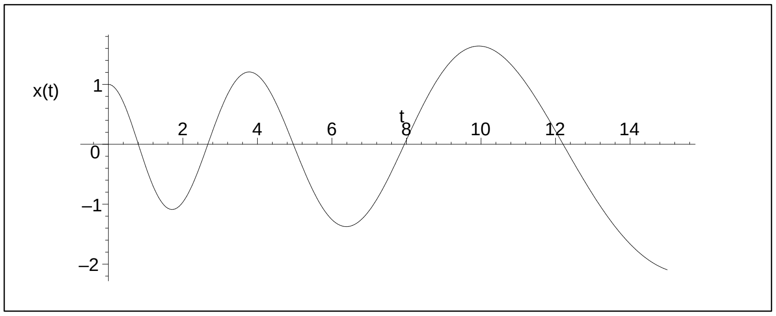

Figure 2: Solution of the IVP in equation 5, using DEplot

Practice Problems:

1. Find a solution of the initial-value problem

$$2x'' + 18e^{-0.5t}x = 0, \quad x(0) = 0, \quad x'(0) = 2$$

   in terms of Bessel functions.

2. Draw a graph of the solution found in #1.

3. Use the MAPLE commands

```
with(DEtools);
de1:= 2.0*diff(x(t),t$2) + 18.0*exp(-0.5*t)*x(t)=0;
DEplot({de1},[x(t)],t=0..15,[[x(0)=0,D(x)(0)=2]],stepsize=0.05);
```

   to draw a numerical approximation to the solution of the IVP in Problem 1.