

## M344 - ADVANCED ENGINEERING MATHEMATICS

### Lecture 7: General solution at a singular point, the Gamma Function

In what follows we will need to have a **general solution** for Bessel's equation; that is, we need two linearly independent solutions  $y_1$  and  $y_2$  such that any solution can be written as a linear combination  $y(t) = c_1y_1(t) + c_2y_2(t)$ . The Method of Frobenius provides this in the following theorem.

**Theorem 1** *Let  $t = 0$  be a regular singular point for the differential equation  $y'' + p(t)y' + q(t)y = 0$ , and let  $r_1$  and  $r_2$  be the roots of the associated indicial equation, where  $\Re(r_1) \geq \Re(r_2)$ . ( $\Re(z)$  is notation for the real part of the complex number  $z$ .) There always exists one series solution of the form  $y_1(t) = t^{r_1} \sum_{n=0}^{\infty} a_n t^n$ , and*

*(i) if  $r_1 - r_2$  is not an integer, there is a second solution of the form  $y_2(t) = t^{r_2} \sum_{n=0}^{\infty} b_n t^n$ ;*

*(ii) if  $r_1 = r_2$ , there is a second solution  $y_2(t) = y_1(t) \ln(t) + \sum_{n=1}^{\infty} b_n t^{n+r_2}$ ;*

*(iii) if  $r_1 - r_2$  is a positive integer, there exists a second solution of the form  $y_2(t) = C y_1(t) \ln(t) + \sum_{n=0}^{\infty} b_n t^{n+r_2}$ , where  $b_0 \neq 0$  and the constant  $C$  may or may not equal 0.*

Before applying this theorem to Bessel's equation, we need to define one more special function called the Gamma function. It will appear in the definition of the second linearly independent solution.

### The Gamma Function

**Def 1** *For any integer  $n > 0$ , the **factorial function**  $n!$  is defined by*

$$n! = 1 \cdot 2 \cdot 3 \cdots n;$$

*that is,  $n!$  is the product of the integers from 1 to  $n$ .*

You have also probably noticed that we defined  $0!$  to be 1. The reason for this will be clear once we define the Gamma function. Unlike the factorial function, the Gamma function is defined for all real  $x$ , except the negative integers, where it becomes infinite. The Gamma function is denoted by the capital Greek letter  $\Gamma$ .

**Def 2** *The **Gamma function** is defined by an improper integral, as follows:*

$$\Gamma(t) = \int_0^{\infty} e^{-u} u^{t-1} du.$$

**Lemma 1** *The Gamma function satisfies the following two properties: (1)  $\Gamma(1) = 1$ , and (2)  $\Gamma(t + 1) = t\Gamma(t)$ .*

**Proof:** To prove the first property,

$$\Gamma(1) = \int_0^{\infty} e^{-u} u^0 du = \int_0^{\infty} e^{-u} du = \lim_{B \rightarrow \infty} \int_0^B e^{-u} du = \lim_{B \rightarrow \infty} (-e^{-B} + e^0) = 1.$$

To prove the second property,

$$\Gamma(t + 1) = \int_0^{\infty} e^{-u} u^{t+1-1} du = \int_0^{\infty} e^{-u} u^t du.$$

Using integration by parts, with  $U = u^t$  and  $dV = e^{-u} du$ , we can write  $\int_0^{\infty} U dV = UV|_0^{\infty} - \int_0^{\infty} V dU$  so that

$$\int_0^{\infty} e^{-u} u^t du = -e^{-u} u^t|_0^{\infty} - \int_0^{\infty} (-e^{-u})(t u^{t-1}) du = 0 + t \int_0^{\infty} e^{-u} u^{t-1} du = t\Gamma(t).$$

Using the above Lemma, we can show that for any positive integer  $n$ ,

$$\boxed{\Gamma(n + 1) = n!}.$$

To see this, write

$$\Gamma(n + 1) = n\Gamma(n) = n(n - 1)\Gamma(n - 1) = \dots = n(n - 1)(n - 2) \dots 1 \cdot \Gamma(1) = n!.$$

Note that this implies that we should define  $0! = \Gamma(0 + 1) = \Gamma(1) = 1$ . The values  $t!$  at the positive integers  $t = n$  are superimposed on the graph of the Gamma function in the Figure below.

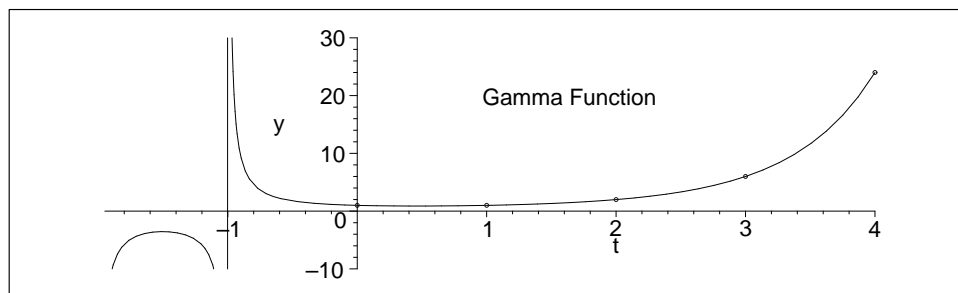


Figure 1: Plot of  $\Gamma(t + 1)$ , with  $t!$  plotted at integers  $t = 0, 1, \dots, 4$

## The second solution of Bessel's equation

We can now find the form of a general solution for Bessel's equation of any order  $\nu$ . If  $2 \cdot \nu$  is not an integer, then the general solution is of the form  $y(t) = c_1 J_\nu(t) + c_2 J_{-\nu}(t)$ , where

$$J_\nu(t) \equiv \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma(1+n+\nu)} \left(\frac{t}{2}\right)^{2n+\nu} \quad \text{and} \quad J_{-\nu}(t) \equiv \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma(1+n-\nu)} \left(\frac{t}{2}\right)^{2n-\nu}.$$

If  $2 \cdot \nu$  is an integer, then the two roots of the indicial equation  $\nu$  and  $-\nu$  differ by an integer; that is,  $\nu - (-\nu) = 2\nu$ . In this case either the second or third form of the second solution is required.

We will be mostly interested in Bessel's equation with  $\nu$  equal to an integer. For any integer  $\nu = n \geq 0$ , the second solution, called the **Bessel function of order  $n$  of the second kind** can be written in the form

$$Y_n(t) \equiv \frac{2}{\pi} \left[ \left( \ln\left(\frac{t}{2}\right) + \gamma \right) J_n(t) + S_1 + S_2 \right].$$

where  $S_1 = -\frac{1}{2} \sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!} \left(\frac{t}{2}\right)^{2k-n}$  and  $S_2 = \frac{1}{2} \sum_{k=0}^{\infty} (-1)^{k+1} \frac{[H(k)+H(k+n)]}{k!(n+k)!} \left(\frac{t}{2}\right)^{2k+n}$ . In the sum  $S_2$ ,  $H(k) \equiv \sum_{k=1}^n \frac{1}{k}$  if  $k > 0$  and  $H(0) = 0$ . The constant  $\gamma$  is Euler's constant, with  $\gamma = \lim_{k \rightarrow \infty} (H(k) - \ln(k)) \approx 0.5772156$ .

Fortunately, we will never have to use this formula to compute  $Y_n(t)$ , since in MAPLE one can simply execute `Bessely(n,t)` to evaluate  $Y_n(t)$  at any value of  $t$ . It is important, however, to recognize that  $Y_n(t) \rightarrow -\infty$  as  $t \rightarrow 0^+$ . Figure (2) shows a graph of the function  $Y_0(t)$  on the interval  $0 \leq t \leq 20$ .

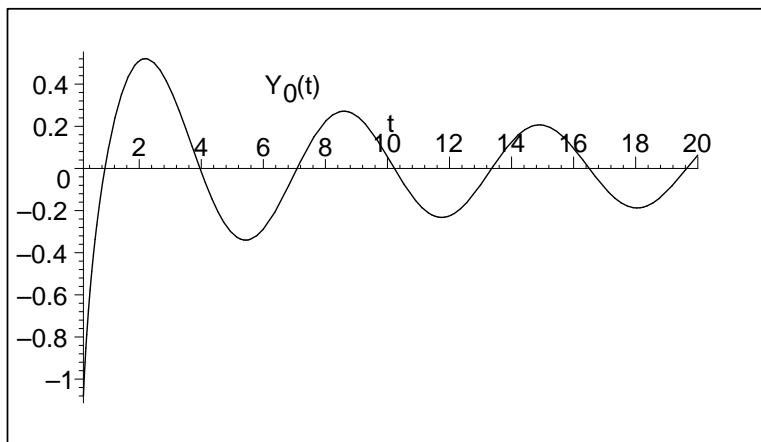


Figure 2: The second solution of Bessel's equation of order 0

To summarize our results on the solution of Bessel's equation: for any integer value  $\nu = n$ , the **general solution of Bessel's equation of order  $n$**  is

$$y(t) = C_1 J_n(t) + C_2 Y_n(t).$$

### Practice Problems:

The starred problems are to be turned in at a date specified in class.

1. Use MAPLE to evaluate  $J_0(t)$ ,  $Y_0(t)$ ,  $J_1(t)$ , and  $Y_1(t)$  at  $t = 20$ .  
Answer:  $J_0(20.0) \approx .16702$ ,  $Y_0(20.0) \approx .06264$ ,  $J_1(20.0) \approx .06683$ ,  $Y_1(20.0) \approx -.16551$ .
2. Find the general solution of the differential equation  $t^2 y'' + t y' + (t^2 - 4)y = 0$  (Note: this is a Bessel equation).
3. \* Use the Gamma function to obtain a formula for the Laplace transform of  $f(t) = t^r$ , for non-integer values of  $r$ . Remember,  $\mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t) dt$ , and for the function  $f(t) = t^r$  it will be a certain Gamma function.
4. \* Airy's equation is  $y'' + xy = 0$ . The substitutions  $u(t) = x^{-\frac{1}{2}}y(x)$  and  $t = \frac{2}{3}x^{\frac{3}{2}}$  converts this equation to  $t^2 u'' + t u' + (t^2 - \frac{1}{9})u = 0$ . Use this to write out the general solution to Airy's equation. **Extra credit:** Show that the above substitutions do convert Airy's equation to the Bessel equation.