M344 - ADVANCED ENGINEERING MATHEMATICS

Lecture 6: Bessel's Equation

Certain differential equations are important because they arise in many different applications. It may be necessary to have a way to compute their solutions quickly, even though these solutions are required in the neighborhood of a singular point. Such a solution, if it can only be defined as a series, rather than in terms of elementary functions, is called a **special function**. Some of the special functions you may run into are Bessel functions, Legendre functions, Laguerre functions, hypergeometric functions, etc. In this section we will study Bessel's equation, and its corresponding solutions called Bessel functions. These special functions will be required when we solve partial differential equations in cylindrical coordinates.

Def 1 *The second-order differential equation*

$$t^2y'' + ty' + (t^2 - \nu^2)y = 0, \tag{1}$$

where ν is a real parameter, is called **Bessel's equation of order** ν.

In standard form, Bessel's equation becomes $y'' + \frac{1}{t}y' + \frac{t^2-\nu^2}{t^2}y = 0$, with $p(t) = \frac{1}{t}$ and $q(t) = \frac{t^2-\nu^2}{t^2}$ both undefined at $t = 0$; therefore, $t = 0$ is a singular point. The limits

$$p_0 = \lim_{t\to 0} tp(t) = \lim_{t\to 0} t\frac{1}{t} = 1 \quad \text{and} \quad q_0 = \lim_{t\to 0} t^2q(t) = \lim_{t\to 0} t^2\frac{t^2 - \nu^2}{t^2} = -\nu^2$$

both exist, so $t = 0$ **is a regular singular point of Bessel's equation**. The indicial equation is

$$r^2 + (p_0 - 1)r + q_0 = r^2 + (1 - 1)r - \nu^2 = r^2 - \nu^2 = 0,$$

with roots $r = \pm\nu$; therefore, there always exists at least one series solution of equation (1) of the form $y(t) = t^\nu \sum_{n=0}^{\infty} a_n t^n$.

In Example 2 of Lecture 5 we found a recursion relation for the coefficients in a series solution of the equation

$$t^2y'' + ty' + (t^2 - \frac{1}{4})y = 0. \tag{2}$$

By definition, this is Bessel's equation of order $\frac{1}{2}$, and we showed that the series solution has the form $t^{\frac{1}{2}} \sum_{n=0}^{\infty} a_n t^n$. In the following example we will find the coefficients in this series.

Example 1 *Find a series solution of the form* $t^{\frac{1}{2}}\sum_{n=0}^{\infty} a_n t^n$ *for the Bessel equation of order* $\frac{1}{2}$.

Letting $y(t) = t^{\frac{1}{2}}\sum_{n=0}^{\infty} a_n t^n = \sum_{n=0}^{\infty} a_n t^{n+\frac{1}{2}}$, *the derivatives are* $y'(t) = \sum_{n=0}^{\infty}(n+\frac{1}{2})a_n t^{n-\frac{1}{2}}$ *and* $y''(t) = \sum_{n=0}^{\infty}(n+\frac{1}{2})(n-\frac{1}{2})a_n t^{n-\frac{3}{2}}$. *Substituting these into equation (2),*

$$t^2\sum_{n=0}^{\infty}(n+\frac{1}{2})(n-\frac{1}{2})a_n t^{n-\frac{3}{2}} + t\sum_{n=0}^{\infty}(n+\frac{1}{2})a_n t^{n-\frac{1}{2}} + (t^2-\frac{1}{4})\sum_{n=0}^{\infty} a_n t^{n+\frac{1}{2}} = 0;$$

and multiplying out the terms,

$$\sum_{n=0}^{\infty}(n+\frac{1}{2})(n-\frac{1}{2})a_n t^{n+\frac{1}{2}} + \sum_{n=0}^{\infty}(n+\frac{1}{2})a_n t^{n+\frac{1}{2}} + \sum_{n=0}^{\infty} a_n t^{n+\frac{5}{2}} - \sum_{n=0}^{\infty}\frac{1}{4}a_n t^{n+\frac{1}{2}} = 0.$$

Making the change of index $m+\frac{1}{2} = n+\frac{5}{2}$ *in the third term, and* $m = n$ *in the other terms,*

$$\sum_{m=0}^{\infty}(m+\frac{1}{2})(m-\frac{1}{2})a_m t^{m+\frac{1}{2}} + \sum_{m=0}^{\infty}(m+\frac{1}{2})a_m t^{m+\frac{1}{2}} + \sum_{m=2}^{\infty} a_{m-2} t^{m+\frac{1}{2}} - \sum_{m=0}^{\infty}\frac{1}{4}a_m t^{m+\frac{1}{2}} = 0.$$

The terms for $m = 0$ *and* $m = 1$ *must be done separately, since the third sum starts at* $m = 2$. *When* $m = 0$, *the coefficient of* $t^{\frac{1}{2}}$ *is*

$$-\frac{1}{4}a_0 + \frac{1}{2}a_0 - \frac{1}{4}a_0 = 0 \cdot a_0 = 0 \ \text{ which implies that } a_0 \text{ is arbitrary.}$$

When $m = 1$,

$$\frac{3}{2}\cdot\frac{1}{2}a_1 + \frac{3}{2}a_1 - \frac{1}{4}a_1 = 2a_1 = 0 \ \text{ implies that } \ a_1 = 0.$$

When $m \geq 2$,

$$(m^2-\frac{1}{4})a_m + (m+\frac{1}{2})a_m + a_{m-2} - \frac{1}{4}a_m = 0 \ \text{ implies that } \ (m^2+m)a_m = -a_{m-2};$$

therefore, the recurrence relation for the coefficients $a_2, a_3, \cdots$ *is*

$$a_m = -\frac{a_{m-2}}{m(m+1)}.$$

With an arbitrary value for a_0, *and* $a_1 = 0$, *we see that all of the odd coefficients* $a_3, a_5, \cdots$ *are zero, and*

$$a_2 = -\frac{a_0}{2\cdot 3} = -\frac{a_0}{3!}, \ \ a_4 = -\frac{a_2}{4\cdot 5} = \frac{a_0}{5!}, \cdots,$$

and in general, $a_{2n} = \frac{a_0}{(2n+1)!}$. *Therefore, the solution of the equation is*

$$y(t) = t^{\frac{1}{2}}(a_0 - \frac{a_0}{3!}t^2 + \frac{a_0}{5!}t^4 - \cdots) = a_0\frac{t^{1/2}}{t}(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \cdots) \equiv a_0\frac{\sin(t)}{\sqrt{t}}.$$

This means that for $\nu = \frac{1}{2}$, one solution of Bessel's equation can be written in terms of elementary functions, but in general this is not the case. In the next example, a series solution of Bessel's equation of order $\nu = 0$ will be found, and we will see that for an integer value of ν, the solution will be a non-elementary function; therefore, the Bessel functions of order ν, for integer values of ν, are special functions.

Example 2 *Find one series solution of Bessel's equation of order* $\nu = 0$*; that is, solve*

$$t^2x'' + tx' + t^2x = 0.$$

We already know that the roots of the indicial equation are both 0. *Therefore, there is one series solution of the form* $x(t) = \sum_{n=0}^{\infty} a_n t^n$. *Notice that in this particular case* $x(t)$ *is an ordinary power series about* $t = 0$. *Substituting* x, x' *and* x'' *into the equation*

$$t^2\sum_{n=0}^{\infty} n(n-1)a_n t^{n-2} + t\sum_{n=0}^{\infty} n a_n t^{n-1} + t^2\sum_{n=0}^{\infty} a_n t^n =$$

$$\sum_{n=0}^{\infty} n(n-1)a_n t^n + \sum_{n=0}^{\infty} n a_n t^n + \sum_{n=0}^{\infty} a_n t^{n+2} = 0.$$

Letting $n + 2 = m$ *in the third sum,*

$$\sum_{m=2}^{\infty} m(m-1)a_m t^m + \sum_{m=1}^{\infty} m a_m t^m + \sum_{m=2}^{\infty} a_{m-2} t^m = 0.$$

Note that the first two sums can start at $m = 2$ *and* $m = 1$*, respectively, since the terms are zero for* $m = 0, 1$ *in the first sum, and for* $m = 0$ *in the second sum. The coefficient of* t *in the entire sum is just* a_1*; therefore,* $a_1 = 0$. *When* $m \geq 2$,

$$m(m-1)a_m + ma_m + a_{m-2} = 0,$$

and the recurrence relation is $a_m = -\frac{a_{m-2}}{m^2}$. *Since* $a_1 = 0$*, this means that all of the odd numbered coefficients are also* 0*, and*

$$a_2 = -\frac{a_0}{2^2}, \quad a_4 = -\frac{a_2}{4^2} = \frac{a_0}{2^2 \cdot 4^2}, \quad \cdots, \quad a_{2m} = \frac{(-1)^m a_0}{2^2 \cdot 4^2 \cdots (2m)^2} = \frac{(-1)^m a_0}{2^{2m}(m!)^2}.$$

The series solution is therefore,

$$x(t) = a_0 - \frac{a_0}{2^2}t^2 + \frac{a_0}{2^2 \cdot 4^2}t^4 + \cdots = a_0 \sum_{n=0}^{\infty} (-1)^n \frac{(t/2)^{2n}}{(n!)^2}.$$

The particular solution $\sum_{n=0}^{\infty}(-1)^n \frac{(t/2)^{2n}}{(n!)^2}$ with $a_0 = 1$ is called the **Bessel function of order** 0 **of the first kind**, and is denoted by $J_0(t)$. Using the same procedure, we can show that for any positive integer ν, there is one solution of Bessel's equation of order ν, of the form

$$J_\nu(t) = \sum_{n=0}^{\infty} (-1)^n \frac{(t/2)^{2n+\nu}}{n!(n+\nu)!},$$

and $J_\nu(t)$ is called the **Bessel function of order** ν **of the first kind**.

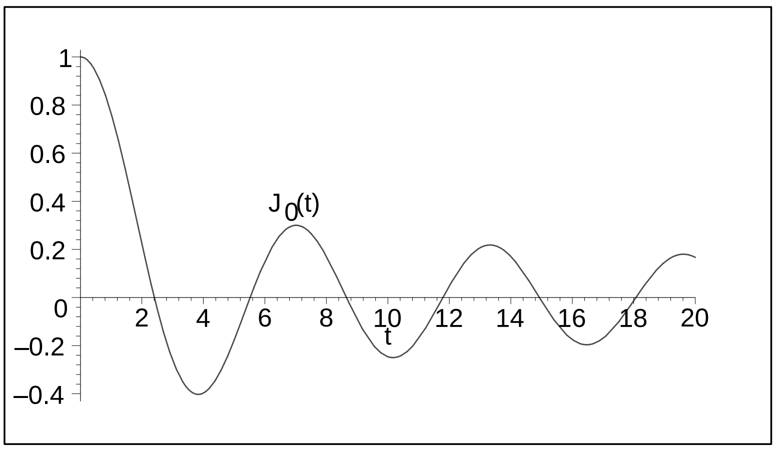

Figure 1:

When we solve partial differential equations, in cylindrical coordinates, we will need to know how the functions J_0 and J_1 behave. It is clear from the series definition that $J_0(0) = 1$ and $J_0'(0) = 0$. In MAPLE, the Bessel function $J_n(t)$ is called $BesselJ(n, t)$, and if the MAPLE command

```
plot(BesselJ(0,t),t = 0..20)
```

is executed, it displays the graph in Figure 1. The function appears to be a damped oscillation. It can be shown analytically, that for any parameter ν, $J_\nu(t)$ approaches the function $\sqrt{\frac{2}{\pi t}}\cos(t - \frac{\nu\pi}{2} - \frac{\pi}{4})$ asymptotically as $t \to \infty$. This implies, for example, that $J_0(t) \to 0$ as $t \to \infty$, and J_0 has infinitely many zeros which get closer and closer to those of $\cos(t - \frac{\pi}{4})$ for large values of t. You will be asked to show in the exercises that $\frac{d}{dx}J_0(x) = -J_1(x)$.

Practice Problems:

1. Find one series solution of Bessel's equation of order 1 by assuming it has the form $y(t) = t\sum_{n=0}^{\infty} a_n t^n$. It should be a constant multiple of $J_1(t)$ defined above.

2. * Write out the first 6 non-zero terms in $J_0(t)$. If you let $z = t^2$, the formula becomes $J_0(z) = \sum_{n=0}^{\infty}(-1)^n \frac{(z)^{2n}}{2^{2n}(n!)^2}$. Use the Ratio Test to show that this series converges for all values of z.

3. * Using MAPLE, compute `BesselJ(0,2.0)` and `BesselJ(0,5.0)`. Then write a loop to sum the series for $J_0(2.0)$ and $J_0(5.0)$ out to $n = 6$. How good is the 6-term series as an approximation to J_0 at t=2.0? at t = 5.0?

4. Show that $J_0'(t) = -J_1(t)$ by showing that the derivative of the series for J_0 is the same as minus the series for J_1.

5. * Use the MAPLE instruction `plot(BesselJ(1,t),t = 0..20.0)` to draw a graph of $J_1(t)$. Describe its general behavior. What happens to $J_1(t)$ as $t \to 0^+$?

6. * We are going to need to know where the function $J_0(t)$ is equal to zero. For n from 0 to 4, execute the MAPLE instruction

   ```
   fsolve(BesselJ(0,t) = 0,t = (n + 0.5) * Pi..(n + 1.0) * Pi);
   ```

   to find the first 5 zeros.