M344 - ADVANCED ENGINEERING MATHEMATICS

Lecture 5: Cauchy-Euler equations and the Method of Frobenius

The next step is to see what can be done with an equation of the form $x''+p(t)x'+q(t)x = 0$ at a point $t = t_0$ where either p or q or both are **not** analytic.

Def 1 *If either $p(t)$ or $q(t)$ or both are not analytic at $t = t_0$, then t_0 is called a* **singular point** *of the equation $x'' + p(t)x' + q(t)x = 0$.*

It may still be possible to find series solutions for $x(t)$. In the following it will be assumed that the singular point t_0 has been moved to $t = 0$. To motivate the method used to solve such an equation, we first examine the simplest equation of this type, the Cauchy-Euler equation.

The Cauchy-Euler Differential Equation

Def 2 *A* **Cauchy-Euler differential equation** *is a second-order linear equation of the form*

$$at^2x''(t) + btx'(t) + cx(t) = 0 \tag{1}$$

where $a, b,$ and c are constants.

In standard form, this equation becomes

$$x''(t) + \frac{1}{t}\frac{b}{a}x'(t) + \frac{1}{t^2}\frac{c}{a}x(t) = 0.$$

It is clear that $t = 0$ is a singular point for this equation. However, if we assume a solution of the form $x(t) = t^n$, then $x'(t) = nt^{n-1}$ and $x''(t) = n(n-1)t^{n-2}$. Substituting these into (1),

$$at^2n(n-1)t^{n-2} + btnt^{n-1} + ct^n = t^n[an(n-1) + bn + c] = 0.$$

If $t \neq 0$, this implies that n must be a root of the quadratic equation

$$an^2 + (b-a)n + c = 0. \tag{2}$$

This quadratic equation is called the **indicial equation**, and its roots are the exponents n for which t^n is a solution of the differential equation (1).

The general solution of the Cauchy-Euler equation (1) depends on the type of roots, and the three cases are given below:

1. If $\boldsymbol{n_1 \neq n_2}$ are two distinct real roots of (2), then $\boldsymbol{x(t) = c_1t^{n_1} + c_2t^{n_2}}$;
2. if **n is a double root** of (2), $\boldsymbol{x(t) = c_1t^n + c_2t^n \ln(t)}$;
3. if $\boldsymbol{n = \alpha \pm \beta \mathbf{i}}$ are two complex conjugate roots of (2), then

$$t^n = e^{n\ln(t)} = e^{(\alpha\pm\beta\mathbf{i})\ln(t)} = (e^{\ln(t)})^\alpha e^{\pm\beta\ln(t)\mathbf{i}}$$
$$= t^\alpha\left(\cos(\beta\ln(t)) \pm \mathbf{i}\sin(\beta\ln(t))\right),$$

where the final result is obtained by using Euler's identity. Then the general solution is $\boldsymbol{x(t) = c_1t^\alpha\cos(\beta\ln(t)) + c_2t^\alpha\sin(\beta\ln(t))}$.

This is all very similar to the solution of the constant coefficient equation $ax''+bx'+cx=0$. Make a note of the similarities, and the differences. The important differences are that the indicial equation is just slightly different than the "characteristic equation" of the constant-coefficient d.e., and that the Cauchy-Euler equation often has solutions which are undefined at $t=0$.

Example 1 *Solve the initial-value problem $4t^2x''+2tx'-6x=0$, with $x(1)=2, x'(1)=\frac{1}{2}$.*

We recognize this as a Cauchy-Euler equation with $a=4, b=2$, and $c=-6$. The indicial equation is $4n^2+(2-4)n-6=4n^2-2n-6=(2n+2)(2n-3)=0$, with two unequal real roots $n_1=-1$ and $n_2=\frac{3}{2}$. This gives the general solution, according to case (1): $x(t)=c_1t^{-1}+c_2t^{\frac{3}{2}}$. Differentiating $x(t)$, $x'(t)=-c_1t^{-2}+\frac{3}{2}c_2t^{\frac{1}{2}}$. Then $x(1)=c_1+c_2=2$ and $x'(1)=-c_1+\frac{3}{2}c_2=\frac{1}{2}$ have simultaneous solution $c_1=c_2=1$; therefore, the solution to the IVP is $x(t)=\frac{1}{t}+t^{\frac{3}{2}}$. Notice that this solution is only defined for $t>0$ and has a vertical asymptote at $t=0$.

Method of Frobenius

The solution of the Cauchy-Euler equation can be used to derive a series method, called the Method of Frobenius, for solving the equation

$$x''+p(t)x'+q(t)x=0 \tag{3}$$

around a singular point $t=0$, if $tp(t)$ and $t^2q(t)$ are both analytic at $t=0$. This means that there exist convergent series $\sum_0^\infty p_nt^n$ and $\sum_0^\infty q_nt^n$ such that $p(t)=\frac{1}{t}\sum_0^\infty p_nt^n$ and $q(t)=\frac{1}{t^2}\sum_0^\infty q_nt^n$.

Def 3 *If $t=0$ is a singular point for equation (3), and $\lim_{t\to 0}tp(t)=p_0$ and $\lim_{t\to 0}t^2q(t)=q_0$ both exist, then $t=0$ is called a* **regular singular point** *for equation (3). If one, or both, of these limits is undefined, $t=0$ is called an* **irregular singular point** *for (3)*

If $t=0$ is a regular singular point for (3), then multiplying (3) by t^2 yields the equation

$$t^2\left(x''+p(t)x'+q(t)x\right)=t^2x''+t^2\left(\frac{1}{t}\sum_0^\infty p_nt^n\right)x'+t^2\left(\frac{1}{t^2}\sum_0^\infty q_nt^n\right)x=$$

$$t^2x''+t(p_0+p_1t+p_2t^2+\cdots)x'+(q_0+q_1t+q_2t^2+\cdots)x=0;$$

and for t close to 0, this acts like the Cauchy-Euler equation $t^2x''+p_0tx'+q_0x=0$. The essential idea behind the Method of Frobenius is to show that if n_1 is the *larger* root of the corresponding indicial equation $n^2+(p_0-1)n+q_0=0$, then equation (3) has a solution of the form

$$t^{n_1}\sum_{k=0}^\infty a_kt^k,$$

where $\sum_{k=0}^\infty a_kt^k$ is a convergent power series about $t=0$. Depending on the exponent n_1, this solution may or may not be defined at $t=0$.

The following two examples illustrate the Method of Frobenius.

Example 2 *Solve the differential equation*

$$t^2x'' + tx' + (t^2 - \frac{1}{4})x = 0. \tag{4}$$

In standard form the equation becomes $x'' + \frac{1}{t}x' + \frac{t^2-1/4}{t^2}x = 0$*, with* $p(t) = \frac{1}{t}$ *and* $q(t) = \frac{t^2-1/4}{t^2}$*. It is clear from this that* $t = 0$ *is a singular point. Calculating the limits*

$$p_0 = \lim_{t\to 0} tp(t) = \lim_{t\to 0} t(\frac{1}{t}) = 1, \qquad q_0 = \lim_{t\to 0} t^2q(t) = \lim_{t\to 0} t^2(\frac{t^2-1/4}{t^2}) = -\frac{1}{4},$$

it can be seen that $t = 0$ *is a regular singular point. The indicial equation is* $r^2 + (1-p_0)r + q_0 = r^2 + (1-1)r - \frac{1}{4} = r^2 - \frac{1}{4} = 0$*, and it has the two unequal real roots* $r = \pm\frac{1}{2}$*. This means that the differential equation has one series solution of the form* $x(t) = t^{1/2}\sum_{n=0}^{\infty} a_nt^n$*. In the exercises you will be asked to find a recurrence relation for the coefficients* a_n*. Equation (4) will be shown in Lecture 6 to be a Bessel Equation.*

Example 3 *Find one series solution for the equation*

$$t^2y''(t) - ty'(t) + (1-t)y(t) = 0 \tag{5}$$

First put the equation into standard form

$$y''(t) - \frac{1}{t}y'(t) + \frac{(1-t)}{t^2}y(t) = 0.$$

This allows us to recognize the coefficient functions $p(t) = -\frac{1}{t}$ *and* $q(t) = \frac{(1-t)}{t^2}$*. Clearly,* $t = 0$ *is a singular point, so we have to compute* $p_0 = \lim_{t\to 0} t(-\frac{1}{t}) = -1$ *and* $q_0 = \lim_{t\to 0} t^2(\frac{(1-t)}{t^2}) = 1$*. Since both limits exist,* $t = 0$ *is a regular singular point.*

The indicial equation $n^2 + (p_0 - 1)n + q_0 = n^2 - 2n + 1 = 0$ *has a double real root* $n = 1$*; therefore the equation has one series solution of the form* $y(t) = t\sum_{n=0}^{\infty} a_nt^n = \sum_{n=0}^{\infty} a_nt^{n+1}$*. If we substitute* $y, y' = \sum_{n=0}^{\infty}(n+1)a_nt^n$ *and* $y'' = \sum_{n=0}^{\infty}(n+1)na_nt^{n-1}$ *into the differential equation (5),*

$$t^2\sum_{n=0}^{\infty}(n+1)na_nt^{n-1} - t\sum_{n=0}^{\infty}(n+1)a_nt^n + (1-t)\sum_{n=0}^{\infty}a_nt^{n+1} = 0$$

can be written as

$$\sum_{n=0}^{\infty}(n+1)na_nt^{n+1} - \sum_{n=0}^{\infty}(n+1)a_nt^{n+1} + \sum_{n=0}^{\infty}a_nt^{n+1} - \sum_{n=0}^{\infty}a_nt^{n+2} = 0.$$

In the final sum let $m + 1 = n + 2$*, so that* $m = n + 1$ *and* $n = m - 1$*; then*

$$\sum_{m=0}^{\infty}(m+1)ma_mt^{m+1} - \sum_{m=0}^{\infty}(m+1)a_mt^{m+1} + \sum_{m=0}^{\infty}a_mt^{m+1} - \sum_{m=1}^{\infty}a_{m-1}t^{m+1} = 0.$$

In this case, since the fourth sum in the above equation has no term when $m = 0$, we need to take the coefficient of t^{0+1} separately; that is, for $m = 0$,

$$(m+1)ma_m - (m+1)a_m + a_m = 1 \cdot 0 \cdot a_0 - 1 \cdot a_0 + a_0 = 0 \cdot a_0 = 0.$$

This implies that a_0 is arbitrary; that is, it can have any value. For any $m = 1, 2, \cdots$, setting the coefficient of t^{m+1} equal to zero gives

$$((m+1)m - (m+1) + 1)a_m = a_{m-1};$$

and this gives the recurrence relation

$$a_m = \frac{a_{m-1}}{m^2}.$$

Computing successive coefficients, $a_1 = \frac{a_0}{1^2} = a_0, a_2 = \frac{a_1}{2^2} = \frac{a_0}{1^2 2^2}$ and in general $a_m = \frac{a_0}{1^2 \cdot 2^2 \cdots m^2} = \frac{a_0}{(m!)^2}$; therefore, one series solution of (5) is

$$y(t) = a_0[t + \frac{t^2}{(1!)^2} + \frac{t^3}{(2!)^2} + \frac{t^4}{(3!)^2} + \cdots] = a_0 \sum_{m=0}^{\infty} \frac{t^{m+1}}{(m!)^2}.$$

The series $\sum_{n=0}^{\infty} \frac{t^{n+1}}{(n!)^2}$ does not represent an elementary function. If MAPLE is used to solve the differential equation (5), it responds with a linear combination of Bessel functions. These will be defined in Lecture 6.

Practice Problems:

The starred problems are to be turned in at a date specified in class.

1. For each equation below determine whether $t = 0$ is an ordinary point, a regular singular point or an irregular singular point:

   (a) $t^2x'' + tx' + 2x = 0$ — Ans: regular singular point

   (b) $t^2(t+1)x'' + x' + x = 0$ — Ans: irregular singular point

   (c) $(t+1)x'' + tx' + 4x = 0$ — Ans: ordinary point

2. For each Cauchy-Euler equation below, find the general solution. Also find the solution of the IVP with $y(1) = 1$ and $y'(1) = 0$. Sketch a graph of $y(t)$ for $0 < t \leq 2$.

   (a) $t^2y'' + 6ty' + 6y = 0$ — Ans: $y(t) = c_1t^{-2} + c_2t^{-3}$

   (b) $4t^2y'' + y = 0$ — Ans: $c_1\sqrt{t} + c_2\sqrt{t}\ln(t)$

   (c)* $t^2y'' - ty' + 17y = 0$

3. * Find the recurrence relation for the coefficients of the series solution of the equation $t^2x'' + tx' + (t^2 - \frac{1}{4})x = 0$ in Example 2. For **extra credit**, show that this series solution can be written in terms of elementary functions.

4. * In Example 3 it was shown that one solution of $t^2y''(t) - ty'(t) + (1-t)y(t) = 0$ is $y(t) = \sum_{n=0}^{\infty} \frac{t^{n+1}}{(n!)^2}$, if we assume that the constant $a_0 = 1$. Use MAPLE to find approximate values of $y(1)$ and $y'(1)$ by summing the appropriate series (use about 10 terms). Then use DEplot to obtain a numerical solution to this differential equation, using the same initial conditions $y(1), y'(1)$. Draw a graph of $\sum_{n=0}^{5} \frac{t^{n+1}}{(n!)^2}$ for $1 \leq t \leq 4$ and compare it to the numerical solution found using the MAPLE DEplot command.