**M344 - ADVANCED ENGINEERING MATHEMATICS**

**Lecture 4: General Solutions, the Aging Spring Equation**

Given any second-order linear differential equation, with non-constant coefficients, there exists a general solution, in terms of series, about any ordinary point $t = t_0$. This is what the following theorem says.

**Theorem 1** *Let $t_0$ be an ordinary point for the equation*

$$y'' + p(t)y' + q(t)y = 0. \tag{1}$$

*Then this equation has two linearly independent analytic solutions of the form $y_1(t) = \sum_{n=0}^{\infty} b_n(t-t_0)^n$ and $y_2(t) = \sum_{n=0}^{\infty} c_n(t-t_0)^n$. The general solution of the equation can be written in the form $y(t) = C_1y_1(t) + C_2y_2(t)$. Moreover, the radius of convergence of the power series solutions is at least as large as the distance from $t_0$ to the nearest singular point (real or complex) of equation (1).*

**Example 1** *Use Theorem 1 to determine the minimum radius of convergence for series solutions (expanded about $t = 0$) for each of the following equations:*
*(a)* $(1-t)y'' + 4ty' + \cos(t)y = 0$;
*(b)* $y'' + \frac{1}{1+t^2}y' + (2+t)y = 0$;
*(c)* $ty'' + \sin(t)y' + t^2y = 0$.

*In (a), $p(t) = \frac{4t}{1-t}$ and $q(t) = \frac{\cos(t)}{1-t}$ are analytic everywhere except at $t = 1$; therefore the minimum radius of convergence of a series solution about $t = 0$ is $R = 1$.*

*In (b) the coefficients are analytic for all real $t$. The coefficient $p(t) = \frac{1}{1+t^2}$ is undefined only at $t = \mathbf{i}$, and the distance from $\mathbf{i}$ to the origin (in the complex plane) is 1. Therefore series solutions expanded about $t = 0$ will converge at least for $-1 < t < 1$.*

*In (c), $p(t) = \sin(t)/t$ and $q(t) = \frac{t^2}{t} = t$ are both analytic for all $t$. To see that $\sin(t)/t$ has no singular points, consider the series $\sin(t)/t = \frac{1}{t}(t - t^3/3! + t^5/5! - \cdots) = (1 - t^2/3! + t^4/5! - \cdots)$, and use the Ratio Test to show that this series converges for all $t$.*

In the next example, we solve completely a non-constant coefficient equation of this type, and show that the two linearly independent solutions may either be elementary functions such as sines, cosines, exponentials, etc., or may be given by a series that cannot be written as a combination of such functions.

**Example 2** *Find a general solution of the differential equation*

$$2y'' + ty' + y = 0. \tag{2}$$

*This equation can be thought of as a mass-spring equation in which the damping coefficient is increasing over time. It starts out undamped, and becomes over damped when $t > \sqrt{8}$. It can be put into standard form by dividing by 2, and it is clear that any value of $t$ is an ordinary point, since $p(t) = t/2$ and $q(t) = 1/2$ are analytic for all $t$. We will assume series solutions around $t = 0$, and according to Theorem 1 these series will converge for all $t$.*

*Let $y(t) = \sum_{n=0}^{\infty} a_n t^n$. Then*

$$2\sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} + t\sum_{n=1}^{\infty} n a_n t^{n-1} + \sum_{n=0}^{\infty} a_n t^n \equiv 0.$$

*Making the change of index $m = n-2$ in the first sum, and $m = n$ elsewhere,*

$$\sum_{m=0}^{\infty} 2(m+2)(m+1)a_{m+2}t^m + \sum_{m=0}^{\infty} m a_m t^m + \sum_{m=0}^{\infty} a_m t^m = 0.$$

$$\sum_{m=0}^{\infty} \left[2(m+2)(m+1)a_{m+2} + m a_m + a_m\right] t^m \equiv 0.$$

*For $m = 0, 1, 2, \cdots$, the coefficient $2(m+2)(m+1)a_{m+2} + (m+1)a_m = 0$; therefore, the recurrence relation is*

$$a_{m+2} = -\frac{a_m}{2(m+2)}.$$

*Let $a_0 = x(0)$ and $a_1 = x'(0)$ be arbitrary constants. Then, solving successively for $a_2, a_3, \cdots$, in terms of $a_0$ and $a_1$,*

$$a_2 = \frac{-a_0}{2\cdot 2}, \quad a_3 = \frac{-a_1}{2\cdot 3}, \quad a_4 = \frac{-a_2}{2\cdot 4} = \frac{a_0}{2^2\cdot 2\cdot 4}, \cdots$$

*and therefore,*

$$x(t) = a_0 + a_1 t - \frac{a_0}{2\cdot 2}t^2 - \frac{a_1}{2\cdot 3}t^3 + \frac{a_0}{2^2\cdot 2\cdot 4}t^4 + \frac{a_1}{2^2\cdot 3\cdot 5}t^5 - \frac{a_0}{2^3\cdot 2\cdot 4\cdot 6}t^6 - \cdots$$

$$= a_0\Big(1 - \frac{t^2}{2\cdot 2} + \frac{t^4}{2^2\cdot 2\cdot 4} - \frac{t^6}{2^3\cdot 2\cdot 4\cdot 6} + \cdots\Big) + a_1\Big(t - \frac{t^3}{2\cdot 3} + \frac{t^5}{2^2\cdot 3\cdot 5} - \frac{t^7}{2^3\cdot 3\cdot 5\cdot 7} + \cdots\Big)$$

*where $a_0$ and $a_1$ are arbitrary. Thus $y(t) = a_0 y_1(t) + a_1 y_2(t)$ where $y_1$ and $y_2$ are the two series solutions referred to in Theorem 1; that is*

$$y_1(t) = 1 - \frac{t^2}{2\cdot 2} + \frac{t^4}{2^2\cdot 2\cdot 4} - \frac{t^6}{2^3\cdot 2\cdot 4\cdot 6} + \cdots + \frac{t^{2n}}{2^n\cdot 2\cdot 4\cdot 6\cdots 2n} + \cdots$$

*and*

$$y_2(t) = t - \frac{t^3}{2\cdot 3} + \frac{t^5}{2^2\cdot 3\cdot 5} - \frac{t^7}{2^3\cdot 3\cdot 5\cdot 7} + \cdots + \frac{t^{2n+1}}{2^n\cdot 3\cdot 5\cdot 7\cdots(2n+1)} + \cdots.$$

*By doing some algebra, the function $y_1$ can be written as an exponential function. Note that the denominator in the nth term is*

$$2^n \cdot 2 \cdot 4 \cdot 6 \cdots 2n = 2^n 2(1)2(2) \cdots 2(n) = 2^{2n} n!;$$

*therefore,* $\frac{t^{2n}}{2^n \cdot 2 \cdot 4 \cdot 6 \cdots 2n} = \frac{(t/2)^{2n}}{n!}$ *and using the Taylor Series for* $e^t$,

$$y_1(t) = \sum_{n=0}^{\infty} (-1)^n \frac{(t/2)^{2n}}{n!} \equiv e^{-(t/2)^2}. \text{ Check it!}$$

*Similarly, the second series, for* $y_2$, *can be simplified by noting that* $1 \cdot 3 \cdot 5 \cdot 7 \cdots (2n+1) = \frac{1 \cdot 2 \cdot 3 \cdots (2n)(2n+1)}{2 \cdot 4 \cdot 6 \cdots (2n)} = \frac{(2n+1)!}{2 \cdot 4 \cdot 6 \cdots 2n} = \frac{(2n+1)!}{2^n n!}$. *Thus,*

$$y_2 = \sum_{n=0}^{\infty} (-1)^n \frac{n!}{(2n+1)!} t^{2n+1},$$

*but this series* **can not** *be written in terms of elementary functions. However, we know from Theorem 1 that it does converge for all* $t$. *This can also be checked by the Ratio Test:*

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{(n+1)!}{(2(n+1)+1)!} \frac{(2n+1)!}{n!} \right| = \lim_{n \to \infty} \left| \frac{(n+1)}{(2n+2)(2n+3)} \right| = 0;$$

*and since* $L = 0$, *the radius of convergence is* $R = \frac{1}{L} = \infty$.

If MAPLE is used to solve the differential equation (2), it responds with the answer $x(t) = C_1 \mathrm{erf}(\mathbf{i}t/2) e^{-t^2/4} + C_2 e^{-t^2/4}$. This is the same as our result, and $\mathrm{erf}(\mathbf{i}t/2)$ is one of the special functions (the complex Error Function) for which MAPLE has a series solution programmed. The point is that you can often solve a non-constant coefficient equation in terms of series, but you must have some idea of what the results look like, so you know how to interpret them. Many of the second-order non-constant linear equations have been given special names, and their series solutions have been studied carefully. We will see examples of this when we study Bessel Functions and other Special Functions later.

### Aging Spring Equation

A slightly more complicated example of an equation with time-varying coefficients is the **Aging Spring equation**

$$mx'' + bx' + ke^{-\nu t} x = 0, \tag{3}$$

which models a mass-spring system where the restoring force of the spring is weakening over time. The new problem here is that the spring constant is replaced by a decaying exponential function, and *series multiplication* must be used to obtain the recurrence relation.

Using the exponential series

$$e^{-\nu t} = 1 - \nu t + \frac{\nu^2}{2!}t^2 + \cdots = \sum_{n=0}^{\infty} \frac{(-\nu t)^n}{n!},$$

and letting $x(t) = \sum_{n=0}^{\infty} a_n t^n$, we can write the term $ke^{-\nu t}x$ in series form as follows:

$$ke^{-\nu t}x = k(\sum_{n=0}^{\infty} \frac{(-\nu t)^n}{n!})(\sum_{n=0}^{\infty} a_n t^n) = k\sum_{n=0}^{\infty} \left(\sum_{j=0}^{n} a_{n-j}\frac{(-\nu)^j}{j!}\right) t^n.$$

The differential equation (3) becomes

$$m\sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} + b\sum_{n=1}^{\infty} na_n t^{n-1} + k\sum_{n=0}^{\infty} \left(\sum_{j=0}^{n} a_{n-j}\frac{(-\nu)^j}{j!}\right) t^n \equiv 0.$$

Changing the index to $s = n-2$ in the first series and $s = n-1$ in the second,

$$\sum_{s=0}^{\infty} m(s+2)(s+1)a_{s+2}t^s + \sum_{s=0}^{\infty} b(s+1)a_{s+1}t^s + \sum_{s=0}^{\infty} k\left(\sum_{j=0}^{s} a_{s-j}\frac{(-\nu)^j}{j!}\right) t^s \equiv 0.$$

Then for $s = 0, 1, 2, \cdots$, each coefficient of $t^s$ must be zero; that is,

$$m(s+2)(s+1)a_{s+2} + b(s+1)a_{s+1} + k\left(\sum_{j=0}^{s} a_{s-j}\frac{(-\nu)^j}{j!}\right) = 0,$$

and the recurrence relation is

$$a_{s+2} = \frac{-b(s+1)a_{s+1} - k\left(\sum_{j=0}^{s} a_{s-j}\frac{(-\nu)^j}{j!}\right)}{m(s+1)(s+2)}, \quad s = 0, 1, 2, \cdots. \qquad (4)$$

In the next example, this recurrence relation will be used to solve an initial-value problem for a particular aging spring equation.

**Example 3** *Find a series solution about $t = 0$ for the IVP*

$$x'' + 0.5x' + 4e^{-0.2t}x = 0, \quad x(0) = 1, \quad x'(0) = 0.$$

*In this particular case, $m = 1, b = 0.5, k = 4, \nu = 0.2$, and the initial conditions give $a_0 = x(0) = 1$ and $a_1 = x'(0) = 0$. Starting with $s = 0$, the recurrence relation above gives*

$$a_2 = \frac{-0.5 \cdot 1 \cdot a_1 - 4\sum_{i=0}^{0} a_{0-i}(-\nu)^i/i!}{1 \cdot 2} = \frac{-4 \cdot a_0 \cdot (-\nu)^0/0!}{2} = -2$$

$$a_3 = \frac{-0.5 \cdot 2 \cdot a_2 - 4\sum_{i=0}^{1} a_{1-i}(-\nu)^i/i!}{2 \cdot 3} = \frac{-(-2) - 4(a_1 - (0.2/1)a_0)}{6} = 7/15;$$

*therefore, the series solution for $x(t)$ begins with the terms*

$$x(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \cdots = 1 - 2t^2 + \frac{7}{15} t^3 + \cdots .$$

*This can be checked in MAPLE by executing the instructions*

$$deq := diff(x(t), t\$2) + 0.5 * diff(x(t), t) + 4.0 * exp(-0.2 * t) * x(t) = 0;$$

$$dsolve(\{deq, x(0) = 1.0, D(x)(0) = 0\}, type = series);$$

*This will produce the response: $x(t) = 1 - 2t^2 + \frac{7}{15}t^3 + \frac{361}{600}t^4 + \frac{6997}{30000}t^5 + \mathcal{O}(t^6)$. The term $\mathcal{O}(t^6)$ means that the rest of the series is "on the order of $t^6$"; that is, it can be considered as some constant times $t^6$ for small $t$ . If more terms in the series are required, you can first execute "Order = N", and this will tell MAPLE to use N terms when it prints out a series.*

The series solution in Example 3 will give a reasonable approximation to $x(t)$ for $0 \le t \le 2$ (see Figure 1), and by using more terms it could be made better, but we will be able to obtain a better series solution, that gives us more information, after we have studied Bessel functions.

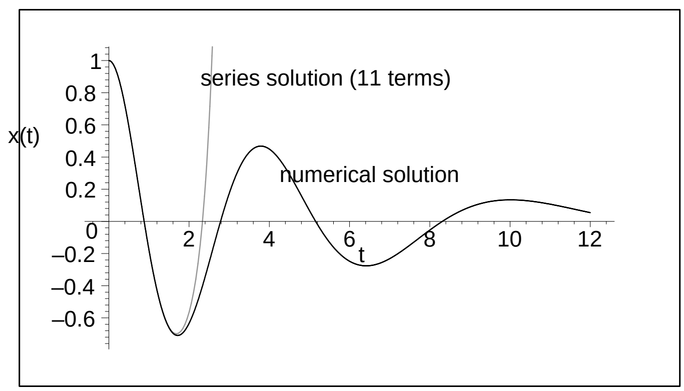


Figure 1: MAPLE and series solutions of $x'' + 0.5x' + 4e^{-0.2t}x = 0$

**Practice Problems:**

1. In Example 2, show by substituting into the equation, that the function $y_1(t) = e^{-t^2/4}$ satisfies the differential equation $2y'' + ty' + y = 0$.

2. For which values of $t > 0$ is the system modelled by the equation $y'' + 3y' + (2+t)y = 0$ under damped?

   Ans: For $t > \frac{1}{4}$.

3. Substitute $y = \sum_0^\infty a_n t^n$ into the equation $y'' + 3y' + (2+t)y = 0$, and find a recurrence relation for the coefficients $a_n$.

   Ans: For $n = 0, 1, \cdots, a_{n+2} = \frac{-3(n+1)a_{n+1} - 2a_n - a_{n-1}}{(n+1)(n+2)}, \quad a_{-1} = 0$.

4. * Find the first 4 non-zero terms in the series about $t = 0$ for the initial-value problem $y'' + 3y' + (2+t)y = 0, \quad y(0) = 1, \quad y'(0) = 0$. Use the recurrence relation found in the previous problem. Check your series using MAPLE. What is the radius of convergence of this series?

5. * Find a general solution of the form $x(t) = a_0 x_1(t) + a_1 x_2(t)$ for the differential equation $y'' + ty' + 4y = 0$. What does MAPLE give for a solution?

6. Multiply the series $y = \sum_0^\infty a_n t^n$ times the Maclaurin series for $e^{-0.1t}$ and write out the first 4 terms (they will contain some of the coefficients $a_0, a_1, \cdots$).

   Ans: $e^{-0.1t} \cdot y(t) = \sum_{n=0}^\infty \frac{(-0.1t)^n}{n!} \cdot \sum_{n=0}^\infty a_n t^n = a_0 + (a_1 - 0.1a_0)t + (a_2 - 0.1a_1 + 0.005a_0)t^2 + (a_3 - 0.1a_2 + 0.005a_1 - \frac{0.001}{6}a_0)t^3 + \cdots$.

7. * Find by hand the first 4 non-zero terms in the series solution of

$$x'' + 2e^{-0.1t}x = 0, \quad x(0) = 2, \quad x'(0) = -1.$$

   Use the recursion relation labelled equation (4). Use MAPLE to check your answer.

8. * Use `DEplot` in MAPLE to solve numerically the initial-value problem $x'' + 2e^{-0.1t}x = 0, \quad x(0) = 2, \quad x'(0) = -1$. Display this solution together with the series solution found in Problem 7.