M344 - ADVANCED ENGINEERING MATHEMATICS

Lecture 3: Series Solutions of Ordinary Differential Equations

You now have the necessary technical tools to solve a mass-spring equation with non-constant coefficients:

$$m(t)x'' + b(t)x' + k(t)x = 0.$$

This equation can be put into what is called **standard form** by dividing by the leading coefficient $m(t)$:

$$x'' + \frac{b(t)}{m(t)}x' + \frac{k(t)}{m(t)}x = \mathbf{x}'' + \mathbf{p}(\mathbf{t})\mathbf{x}' + \mathbf{q}(\mathbf{t})\mathbf{x} = \mathbf{0}. \tag{1}$$

A value of $t = t_0$ is called an **ordinary point** for equation (1) if **both** $p(t)$ and $q(t)$ are analytic at $t = t_0$. Remember that this means p and q both have convergent power series in some interval around t_0. Any point t where either p or q, or both, are not analytic is called a **singular point** of equation (1).

Example 1 *Find all singular points of the differential equation* $t(t-2)x'' + \sin(t)x' + \frac{1}{t+3}x = 0$. *First put the equation into standard form:*

$$x'' + \frac{\sin(t)}{t(t-2)}x' + \frac{1}{(t+3)t(t-2)}x = 0.$$

Then the singular points are $t = 0, 2,$ *and* -3*; that is, at the zeros of the denominators of* p *and* q*. Every other value of* t *is an ordinary point.*

Our method for solving equations with non-constant coefficients involves assuming a series solution $x(t)$ around some initial point $t = t_0$. We can always make a change of independent variable, letting $\bar{t} = t - t_0$, so that all series are expanded around $\bar{t} = 0$ and the Taylor series for x is then a Maclaurin series. *In this lecture we will assume that this has been done.*

If $t = 0$ is an ordinary point for (1), then p and q are both analytic at $t = 0$ (that is, have convergent Maclaurin series); and we can write $p(t) = \sum_0^\infty p_n t^n$ and $q(t) = \sum_0^\infty q_n t^n$. Assume a solution of (1) has the form

$$x(t) = \sum_0^\infty a_n t^n.$$

Note that if initial conditions $x(0) = x_0$ and $x'(0) = v_0$ are specified, then

$$x(t) = x_0 + v_0 t + a_2 t^2 + a_3 t^3 + \cdots .$$

That is, in the Maclaurin series for $x(t)$ the first two coefficients are given by the initial conditions, $a_0 = x(0)$ and $a_1 = x'(0)$. Do you see why?

By differentiating the series for $x(t)$, the series for x' and x'' are:

$$x'(t) = \sum_{n=1}^{\infty} n a_n t^{n-1} \quad \text{and} \quad x''(t) = \sum_{n=2}^{\infty} n(n-1) a_n t^{n-2}.$$

Substituting these into equation (1),

$$\sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} + \sum_{n=0}^{\infty} p_n t^n \cdot \sum_{n=1}^{\infty} n a_n t^{n-1} + \sum_{n=0}^{\infty} q_n t^n \cdot \sum_{n=0}^{\infty} a_n t^n \equiv 0. \qquad (2)$$

By multiplying series and combining coefficients of like powers of t in the three resulting series, this last equation can be written as a single power series

$$\sum_{n=0}^{\infty} C_n t^n \equiv 0, \qquad (3)$$

To make the series in (3) equal zero for all values of t, *each* coefficient C_n must be zero (this is a mathematical theorem). We will see that this gives us what is called a **recurrence relation** for the coefficients a_n of the unknown function $x(t) = \sum_{n=0}^{\infty} a_n t^n$.

Changing the Index of Summation

In order to write (2) in the form (3) it is usually necessary to change one or more index of summation so that all three series that are to be added together are in terms of powers t^m. For example, by making the substitution $m = n+k$, and therefore $n = m-k$, we can write

$$\sum_{n=0}^{\infty} c_n t^{n+k} \equiv \sum_{m=k}^{\infty} c_{m-k} t^m = c_0 t^k + c_1 t^{k+1} + \cdots .$$

Note that the sum is exactly the same in both cases, and only the index of summation, and its starting value, have been changed.

Example 2 *Change the index of summation so that $3t^2 \sum_{n=0}^{\infty} a_n t^{n-1}$ is of the form (3).*

First, multiply each term in the series by $3t^2$:

$$3t^2 \sum_{n=0}^{\infty} a_n t^{n-1} \equiv \sum_{n=0}^{\infty} 3a_n t^{n+1}.$$

Then, letting $m = n+1$ and, therefore, $n = m-1$,

$$\sum_{n=0}^{\infty} 3a_n t^{n+1} \equiv \sum_{m=1}^{\infty} 3a_{m-1} t^m.$$

Note that the index m now starts at $m = 1$, since n started at 0, and $m = n+1$. Check that this last sum gives exactly the same terms as $3t^2 \sum_{n=0}^{\infty} a_n t^{n-1}$.

We are now ready to solve an equation with non-constant coefficients.

Example 3 *Solve the initial-value problem*

$$x'' + (1+t)x' + 2x = 0, \;\; x(0) = 1, \;\; x'(0) = 0. \tag{4}$$

Note that this equation can be thought of as a model of a mass-spring system with mass 1, spring constant 2, and damping coefficient $b(t) \equiv 1+t$ increasing from 1 to ∞ as t increases from 0 to ∞. This is a system which is under damped at time $t = 0$ and becomes over damped after t reaches $\sqrt{8} - 1 \approx 1.83$ seconds. You should think about what the solution will look like.

The differential equation is already in standard form, with $p(t) = 1+t$ and $q(t) = 2$. These functions are polynomials (finite power series), hence analytic at $t = 0$. Assuming a solution $x(t) = \sum_0^\infty a_n t^n$, substitution into equation (4) gives

$$\sum_{n=0}^{\infty} n(n-1)a_n t^{n-2} + (1+t) \cdot \sum_{n=0}^{\infty} n a_n t^{n-1} + 2\sum_{n=0}^{\infty} a_n t^n \equiv 0. \tag{5}$$

We have let the indices on the sums for x' and x'' start at zero, but notice that in the series for x'' the first two terms are zero, and in the series for x' the first term is zero.

We first multiply the polynomial $(1+t)$ times the series in the second term:

$$(1+t)\cdot\left(\sum_{n=0}^{\infty} n a_n t^{n-1}\right) = \sum_{n=0}^{\infty} n a_n t^{n-1} + t\cdot\sum_{n=0}^{\infty} n a_n t^{n-1} = \sum_{n=0}^{\infty} n a_n t^{n-1} + \sum_{n=0}^{\infty} n a_n t^n.$$

Then equation (5) can be written as

$$\sum_{n=0}^{\infty} n(n-1)a_n t^{n-2} + \sum_{n=0}^{\infty} n a_n t^{n-1} + \sum_{n=0}^{\infty} n a_n t^n + 2\sum_{n=0}^{\infty} a_n t^n \equiv 0. \tag{6}$$

To put this into the form of equation (3), we need to make each of the series be of the form $\sum c_m t^m$, by making an appropriate change of index.

The index in the first term can start at $n = 2$, since the terms for $n = 0$ and $n = 1$ are both 0. Then, letting $m = n-2, n = m+2$, m can start at 0 and the series can be rewritten as

$$\sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} \equiv \sum_{m=0}^{\infty} (m+2)(m+1)a_{m+2} t^m.$$

Similarly, in the second series of (6) the term for $n = 0$ is zero so we can start at $n = 1$. Making the change of index $m = n-1, n = m+1$, the second term becomes

$$\sum_{n=1}^{\infty} n a_n t^{n-1} \equiv \sum_{m=0}^{\infty} (m+1)a_{m+1} t^m.$$

In the final two series in equation (6) we just let $m = n$.

Now equation (6) can be written as

$$\sum_{m=0}^{\infty}(m+2)(m+1)a_{m+2}t^m + \sum_{m=0}^{\infty}(m+1)a_{m+1}t^m + \sum_{m=0}^{\infty} ma_m t^m + \sum_{m=0}^{\infty} 2a_m t^m$$

$$\equiv \sum_{m=0}^{\infty}\left[(m+2)(m+1)a_{m+2} + (m+1)a_{m+1} + ma_m + 2a_m\right]t^m \equiv \sum_{m=0}^{\infty} C_m t^m \equiv 0.$$

For each $m = 0, 1, 2, \cdots$, we must have

$$C_m = (m+2)(m+1)a_{m+2} + (m+1)a_{m+1} + (m+2)a_m = 0,$$

and this can be solved for a_{m+2}, the coefficient with the largest index:

$$a_{m+2} = \frac{-(m+1)a_{m+1} - (m+2)a_m}{(m+1)(m+2)}.$$

This gives us the **recurrence relation** *for the coefficients a_m. It is a formula that allows you to find the value of a_{m+2} once the values of $a_0, a_1, \cdots, a_{m+1}$ are known.*

Since we already know that $a_0 = x(0) = 1$ and $a_1 = x'(0) = 0$, this allows us to compute as many more coefficients as desired. For example, the next three coefficients are found by letting m have the values $0, 1$, and 2:

$$m = 0, \qquad a_2 = \frac{-a_1 - 2a_0}{1 \cdot 2} = \frac{0 - 2 \cdot 1}{1 \cdot 2} = -1;$$

$$m = 1, \qquad a_3 = \frac{-2a_2 - 3a_1}{2 \cdot 3} = \frac{2 - 0}{2 \cdot 3} = \frac{1}{3};$$

$$m = 2, \qquad a_4 = \frac{-3a_3 - 4a_2}{3 \cdot 4} = \frac{-3 \cdot \frac{1}{3} - 4(-1)}{3 \cdot 4} = \frac{1}{4}.$$

Therefore, the first few terms in the series for $x(t)$ are

$$x(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + \cdots = 1 - t^2 + \frac{1}{3}t^3 + \frac{1}{4}t^4 + \cdots.$$

These terms can be checked by using the MAPLE instructions

```
de1:=diff(x(t),t$2)+(1+t)*diff(x(t),t)+2x(t)=0;

dsolve({de1,x(0)=1,D(x)(0)=0},x(t),type=series);
```

This produces the output

`x(t) =` $1 - 1 * t^2 + 1/3 * t^3 + 1/4 * t^4 - 2/15 * t^5 + O(t^6)$

Note: to see more than 6 terms in the series, you can first execute the command `Order:=n:` *, where n can be any positive integer.*

Figure (1) shows a MAPLE plot, computed using the instructions

```
with(plots) :

P1 := DEplot ({de1}, [x(t)], t = 0..4, [[x(0) = 1.0, D(x)(0) = 0]],

        stepsize = 0.05, linecolor = BLACK) :

P2 := plot({series - for - x(t)}, t = 0..4) :

display(P1, P2);
```

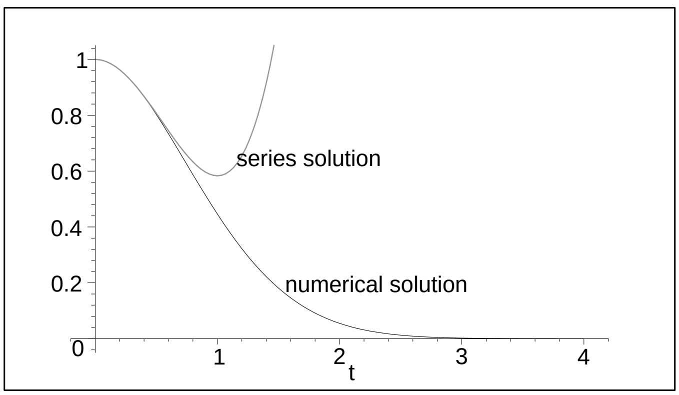

Figure 1: MAPLE and series solutions of $x'' + (1 + t)x' + 2x = 0$

MAPLE uses an adaptive Runge-Kutta method to generate a numerical solution of the initial-value problem. The series solution is also shown on the same graph, and it stays close to the numerical solution only for t between 0 and 0.7. More terms could be used in the series to get a better solution, but in general the numerical solution will be more accurate for *large* values of t. However, the series solution has certain advantages. It gives an analytic formula for $x(t)$ near $t = 0$ and can be used to show the dependence of the solution on parameters, whereas the numerical solution cannot.

Practice Problems:

1. For each of the following equations, determine all ordinary points (remember to first put the equation into standard form):

   (a) $tx'' + (1-t)x' + 2x = 0$ Ans: All t except $t = 0$

   (b) $x'' + (\sqrt{t-3})x' + t^2 x = 0$ Ans: All $t \geq 3$

   (c) $(2+t^2)x'' + \sin(t)x' + (5+2t^3)x = 0$ Ans: $t \in (-\infty, \infty)$

   (d)* $t(1-t)x'' + (t^2-t)x' + 2tx = 0$

2. In each of the following sums, change the index of summation so the sum is of the form $\sum_{m=m_0}^{\infty} c_m t^m$. Also, determine the initial index m_0.

   (a) $\sum_{n=2}^{\infty} n(n-1)a_n t^{n-2}$ Ans: $\sum_{m=0}^{\infty}(m+2)(m+1)a_{m+2}t^m$

   (b) $t^2 \sum_{n=0}^{\infty} a_n t^{n-1}$ Ans: $\sum_{m=1}^{\infty} a_{m-1}t^m$

   (c) $\sum_{n=1}^{\infty} a_{n+2}t^{n+2}$ Ans: $\sum_{m=3}^{\infty} a_m t^m$

3. In each of the following equations, assume $x(t) = \sum_{n=0}^{\infty} a_n t^n$ and find a recurrence relation for the a_n.

   (a) $x'' + tx' + 4x = 0$ Ans: $a_{m+2} = \frac{-(m+4)a_m}{(m+1)(m+2)}, m = 0, 1, \cdots$

   (b) $x''+2x'+(1+t)x = 0$ Ans: $a_{m+2} = \frac{-2((m+1)a_{m+1}-a_m-a_{m-1}}{(m+2)(m+1)}(a_{-1} = 0)$

   (c)* $2x'' + (t-1)x' + x = 0$

4. * Use the recurrence relation found in 3(b), and write out the first 5 nonzero terms in the series solution of $x'' + 2x' + (1+t)x = 0$. Assume the initial conditions are $x(0) = 1$ and $x'(0) = 0$. Check your answer by using the MAPLE commands

```
de2 := diff(x(t),t$2) + 2 * diff(x(t),t) + (1 + t) * x(t) = 0;

dsolve ({de2, x(0) = 1, D(x)(0) = 0} , [x(t)], type = series) ;
```