M344 - ADVANCED ENGINEERING MATHEMATICS

Lecture 23: A PDE Model Used in Cooking

The following very interesting partial differential equation was brought to my attention by Steve Gifford. It appears in a paper found on the Web at

$$\underline{\texttt{http : //amath.colorado.edu/ baldwind/sous} - \texttt{vide.html}}.$$

In this paper, a method of cooking called "Sous Vide" is described. It involves boiling things at a fixed temperature in sealed plastic bags for possibly a very long time. The temperature T of the food in the bag is assumed to satisfy the following version of the one-dimensional heat equation:

$$T_t = \alpha(T_{rr} + \frac{\beta}{r}T_r), \quad T(r,0) = T_0,$$

with boundary conditions

$$T_r(0,t) = 0, \quad kT_r(R,t) = h\left(T_{water} - T(R,t)\right);$$

where $0 \leq r \leq R$ is the distance from the center of the bag, $t \geq 0$ is time, and $0 \leq \beta \leq 2$ is a geometric factor that makes it possible to adjust for a bag of arbitrary shape, from a large slab ($\beta = 0$) to a long cylinder ($\beta = 1$) to a sphere ($\beta = 2$). T_0 is the initial temperature of the food in the bag, and T_{water} is the constant temperature of the water bath in which it is immersed.

If we let the function we are solving for be $U(r,t) = T(r,t) - T_{water}$, then the problem in terms of U becomes:

$$\boxed{\begin{gathered} U_t = \alpha\{U_{rr} + \frac{\beta}{r}U_r\}, \quad U(r,0) = T(r,0) - T_{water} = T_0 - T_{water} \\[2ex] U_r(0,t) = 0, \quad U_r(R,t) + \frac{h}{k}U(R,t) = 0, \end{gathered}}$$

with homogeneous boundary conditions.

This is much like the one-dimensional heat equation we solved, except that it is expressed in cylindrical coordinates, and the parameter β multiplies the term $\frac{1}{r}U_r$ in the Laplacian.

Letting $U(r,t) = X(r)Y(t)$,

$$XY' = \alpha\{X''Y + \frac{\beta}{r}X'Y\}.$$

The variables can be separated by dividing both sides of this equation by αXY:

$$\frac{Y'}{\alpha Y} = \frac{X''}{X} + \frac{\beta}{r}\frac{X'}{X} = -\lambda(\text{constant}),$$

and the two ordinary differential equations in X and Y are

$$\boxed{Y' = -\alpha\lambda Y \text{ and } X'' + \tfrac{\beta}{r}X' + \lambda X = 0.}$$

If the equation

$$X'' + \frac{\beta}{r}X' + \lambda X = 0 \tag{1}$$

is multiplied by r^β, it can be seen to be a **Sturm-Liouville equation**

$$r^\beta X'' + r^\beta \frac{\beta}{r}X' + r^\beta \lambda X \equiv \frac{d}{dr}(r^\beta X') + \lambda r^\beta X = 0$$

with weight factor $w(r) = r^\beta$; therefore, if the eigenvalues λ_n and corresponding eigenfunctions $X_n(r)$ are found, we know that the family of functions $\{X_n(r)\}_{n=1}^{\infty}$ is an orthogonal family on $0 \le r \le R$. This means that

$$\boxed{\int_0^R r^\beta X_j(r)X_k(r)dr = 0 \text{ whenever } j \neq k.}$$

To solve equation (1), make the substitutions $t = \sqrt{\lambda}r$ and $Z(t) = X(r)$. Then (1) becomes

$$\lambda(Z''(t) + \frac{\beta}{t}Z'(t) + Z(t)) = 0.$$

This is not quite Bessel's equation of order 0, because of the parameter β; but using MAPLE, the general solution is found to be

$$Z(t) = C_1 t^{\frac{1}{2}(1-\beta)} J_{\frac{1}{2}(\beta-1)}(t) + C_2 t^{\frac{1}{2}(1-\beta)} Y_{\frac{1}{2}(\beta-1)}(t).$$

As $t \to 0^+$, the function $Y_\nu(t)$ tends to $-\infty$ for any order ν, so in order to make the temperature finite at $r = 0$, it is necessary to set $C_2 = 0$. Thus the required solution of equation (1) is any constant multiple of

$$\boxed{X(r) = Z(t) = Z(\sqrt{\lambda}r) = (\sqrt{\lambda}r)^{\frac{1}{2}(1-\beta)} J_{\frac{1}{2}(\beta-1)}(\sqrt{\lambda}r).}$$

The two boundary conditions on the temperature function $U(r,t)$ can be used to find boundary conditions on $X(r)$, as follows:

$$U_r(0,t) = X'(0)Y(t) = 0, \quad t > 0 \Rightarrow X'(0) = 0,$$

$$U_r(R,t) + \frac{h}{k}U(R,t) = 0, \quad t > 0 \Rightarrow X'(R) + \frac{h}{k}X(R) = 0.$$

To find the eigenvalues λ_n, we need to make $X(r)$ satisfy the two boundary conditions. Define $p = \frac{1}{2}(\beta - 1)$ and $x = \sqrt{\lambda}r$. Then we can use the known series for the Bessel function J_p to write

$$X(r) = x^{-p}J_p(x) = x^{-p}\sum_{n=0}^{\infty}\frac{(-1)^n(\frac{x}{2})^{2n+p}}{n!\Gamma(1+n+p)} = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{2^{2n+p}n!\Gamma(1+n+p)},$$

and since this is a Taylor series in powers of x^2, it can be seen that $X'(0) = 0$ as required. The series can also be used to compute $X(0) = \frac{1}{2^p\Gamma(1+p)}$.

The second boundary condition requires that $X'(R)/X(R) = -\frac{h}{k}$. To find the derivative of the function $X(r)$, we can use a known formula which states that for any order p, $J_p'(x) = \frac{1}{2}\left(J_{p-1}(x) - J_{p+1}(x)\right)$. Using this, with $x = \sqrt{\lambda}r$ and $p = \frac{1}{2}(\beta - 1)$, the chain rule for differentiation implies that

$$X'(r) = \frac{d}{dr}\left(x^{-p}J_p(x)\right) = \frac{d}{dx}\left(x^{-p}J_p(x)\right)\frac{dx}{dr}.$$

Therefore, by the product rule,

$$X'(r) = \left(-px^{-p-1}J_p(x) + x^{-p}\left(\frac{J_{p-1}(x) - J_{p+1}(x)}{2}\right)\right)\sqrt{\lambda}$$

and

$$\frac{X'(r)}{X(r)} = \frac{X'(r)}{x^{-p}J_p(x)} = \left(\frac{-p}{x} + \frac{1}{2}\left(\frac{J_{p-1}(x) - J_{p+1}(x)}{J_p(x)}\right)\right)\sqrt{\lambda}.$$

We can then set

$$\frac{X'(R)}{X(R)} = \left(\frac{-p}{\sqrt{\lambda}R} + \frac{1}{2}\left(\frac{J_{p-1}(\sqrt{\lambda}R) - J_{p+1}(\sqrt{\lambda}R)}{J_p(\sqrt{\lambda}R)}\right)\right)\sqrt{\lambda} = -\frac{h}{k},$$

and with a little algebra it can be seen that

$$\frac{1}{2}\left(\frac{J_{p-1}(\sqrt{\lambda}R) - J_{p+1}(\sqrt{\lambda}R)}{J_p(\sqrt{\lambda}R)}\right) = \left(\frac{-h}{k\sqrt{\lambda}} + \frac{p}{\sqrt{\lambda}R}\right) = \frac{p}{\sqrt{\lambda}R} - \frac{hR}{k\sqrt{\lambda}R}.$$

This means that we must find values $z \equiv \sqrt{\lambda}R$ such that

$$\frac{1}{2}\left(\frac{J_{p-1}(z) - J_{p+1}(z)}{J_p(z)}\right) = \left(p - \frac{h}{k}R\right)\frac{1}{z}.$$

The function $\frac{J_{p-1}(z) - J_{p+1}(z)}{J_p(z)}$ has vertical asymptotes at the zeros of $J_p(z)$, and these are known to approach the zeros of $\cos(z - \frac{p}{2}\pi - \frac{\pi}{4})$ as $z \to \infty$; therefore, there will be one intersection z_n between $(n - \frac{3}{4} + \frac{p}{2})\pi$ and $(n + \frac{1}{4} + \frac{p}{2})\pi$ for each integer $n = 1, 2, \cdots$. These intersections can be found using the MAPLE command `fsolve`. The graph in Figure 1 shows the first 5 intersections, using the parameter values in Example 1 below.

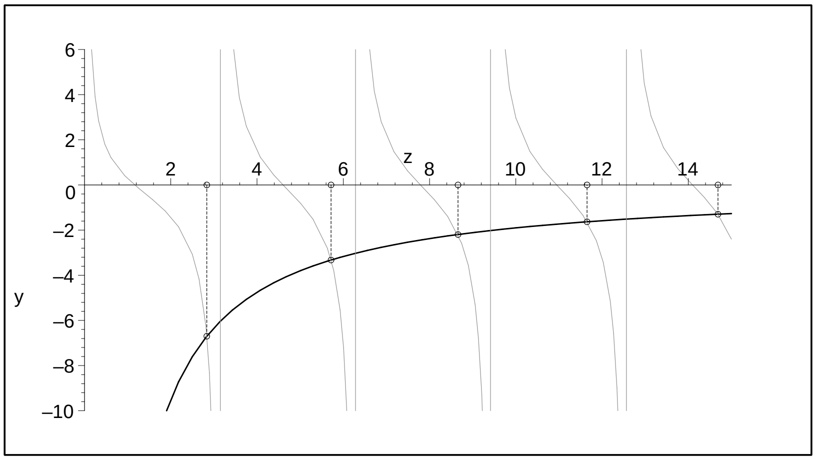


Figure 1: Intersections $z_1 \cdots z_5$ with $\beta = 2$

Once the z_n are found, we can set $\lambda_n = (\frac{z_n}{R})^2$. The functions $Y_n(t)$ can then be found by solving $Y_n'(t) = -\alpha\lambda_n Y_n(t)$. This first-order differential equation has solution $Y_n(t) = e^{-\alpha\lambda_n t}$, and the product $U_n(r,t) = X_n(r)Y_n(t)$ is a solution of the partial differential equation for each integer $n = 1, 2, \cdots$. Since the pde is linear, its general solution can be written in the form of an infinite series

$$U(r,t) = \sum_{n=1}^{\infty} A_n U_n(r,t) = \sum_{n=1}^{\infty} A_n e^{-\alpha\lambda_n t}(\sqrt{\lambda_n}r)^{-p} J_p(\sqrt{\lambda_n}r).$$

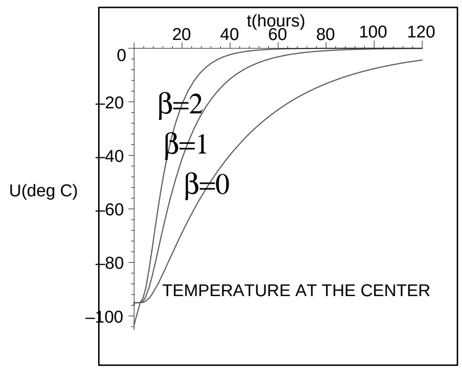
The initial condition on $U(r,t)$ required that

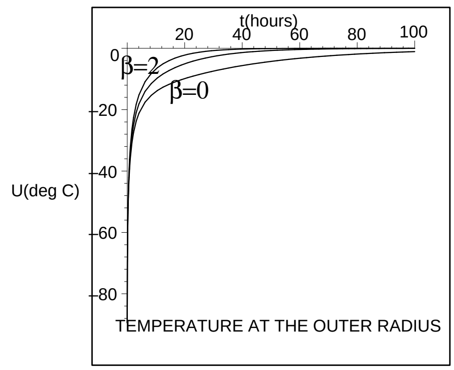
$$U(r,0) = \sum_{n=1}^{\infty} A_n X_n(r) = \sum_{n=1}^{\infty} A_n(\sqrt{\lambda_n}r)^{-p} J_p(\sqrt{\lambda_n}r) = T_0 - T_{water}.$$

Since the functions $X_n(r)$ form an orthogonal set on $[0, R]$ with weight function r^β, the coefficients are

$$A_n = \frac{\int_0^R U(r,0)X_n(r)r^\beta dr}{\int_0^R (X_n(r))^2 r^\beta dr} = \frac{\int_0^R (T_0 - T_{water})(\sqrt{\lambda_n}r)^{-p} J_p(\sqrt{\lambda_n}r) r^\beta dr}{\int_0^R ((\sqrt{\lambda_n}r)^{-p} J_p(\sqrt{\lambda_n}r))^2 r^\beta dr}.$$

Example 1 *For a representative set of values of the parameters, compare the time it takes for the temperature in the center of the food package to rise close to the temperature of the water bath for $\beta = 0, 1,$ and 2.*





The graphs in the above figure were generated by the MAPLE program shown below. Realistic values for the parameters were taken from the web article cited on page 1 of this lecture. The value of the heat transfer coefficient $h = 25W/m^2K$, the value of the thermal conductivity of the food $k = 0.5W/mK$, and the diameter of the package was chosen to be $R = 0.2m$. Note that the constant $p - \frac{h}{k}R$ is dimensionless, since R is in meters. To have time t in hours, the thermal diffusivity $\alpha = 1.5 \times 10^{-7} m^2/sec$ was multiplied by 3600.

SERIES SOLUTION FOR U(r,t)

```
> h:=25:  k:=0.5:  alpha:=1.5E-7*3600:  R:=0.2:  T0:=5:  Twater:=100:
> beta:=2; p:=(beta-1.0)/2.0; Nterms:=50;
> for n from 1 to Nterms do
> z[n]:=fsolve((BesselJ(p-1,z)-BesselJ(p+1,z))/BesselJ(p,z)=
       2.0*(p-h*R/k)/z,z=(n-0.75+p/2.0)*Pi..(n+0.25+p/2.0)*Pi);
> lam[n]:=(z[n]/R)∧2;
> A[n]:=int((T0-Twater)*(z[n]*r/R)∧(-p)*BesselJ(p,z[n]*r/R)*r∧(beta),r=0..R)
       /int(((z[n]*r/R)∧(-p)*BesselJ(p,z[n]*r/R))∧2*r∧(beta),r=0..R); od:
> U:=proc(r,t) local S; global A, lam, alpha, p, z, R, Nterms;
> if r=0 then
     S:=sum(A[j]*exp(-alpha*lam[j]*t),j=1..Nterms)/(2.0∧p*GAMMA(p+1.0))
> else S:=sum(A[j]*exp(-alpha*lam[j]*t)*(z[j]*r/R)∧(-p))
      *BesselJ(p,z[j]*r/R),j=1..Nterms); fi; S:
> end proc;
U(0,20); Ans = -20.812385
```

For small values of t, the series for U converges very slowly, and it is useful to compare values of $U(r, t)$ obtained by this method with those obtained using a numerical method. Problem 5 below will ask you to do this.

Exercises:

1. Using the parameter values from Example 1, run the MAPLE program to find the series solution $U(r,t)$. For each value $\beta = 0, 1$, and 2, compute the values $U(0,10)$, $U(0,20)$, $U(0,60)$ and $U(0,120)$.

2. The equation $U_t = \alpha\{U_{rr} + \frac{\beta}{r}U_r\}$ can be approximated by the difference equation

$$\frac{U(r,t+\Delta t) - U(r,t)}{\Delta t} = \alpha\left[\frac{U(r+\Delta r,t) - 2U(r,t) + U(r-\Delta r,t)}{(\Delta r)^2} + \frac{\beta}{r}\left(\frac{U(r+\Delta r,t) - U(r,t)}{\Delta r}\right)\right].$$

   Solve this equation for $U(r, t + \Delta t)$ in terms of values of U at time t.

3. How would you express the two boundary conditions $U_r(0,t) = 0$ and $U_r(R,t) + \frac{h}{k}U(R,t) = 0$ as difference formulas in U?

4. Check that the MAPLE program below solves the difference equation in Problem 2 with the boundary conditions given in Problem 3. Explain how the program handles the two boundary conditions.

NUMERICAL SOLUTION FOR U(r,t)

```
> h:=25:  k:=0.5:  alpha:=1.5E-7*3600:  R:=0.2:  T0:=5:  Twater:=100:
> beta:=2:
> N:=20:  J:=2400:  dr:=R/N: dt:=120/J:
       C:=alpha*dt/dr^2; C2:=beta*dr/2.0:  C3:=2.0*h*dr/k:
> for i from 1 to N+2 do u[i-1,0]:=T0-Twater; od:
> for j from 0 to J-1 do
> for i from 1 to N do
> u[i,j+1]:=u[i,j]+C*(u[i+1,j]-2*u[i,j]+u[i-1,j]
       +(C2/(i*dr))*(u[i+1,j]-u[i-1,j]));
> od;
> u[0,j+1]:=u[1,j+1];
> u[N+1,j+1]:=u[N-1,j+1]-C3*u[N,j+1];
> od:
u[0,400]; Ans=-20.766106
```

5. Choose appropriate values for N (number of intervals in the partition of the r-axis) and J (number of time steps), and compute the numerical solution $u[i,j]$ on the interval $0 \le t \le 120$ hours. In order for the numerical method to be *stable*, the constant $C = \alpha\frac{dt}{dr^2}$ must be less than 0.5. For each $\beta = 0, 1$, and 2, compare the values you get for $U(0,10)$, $U(0,20)$, $U(0,60)$ and $U(0,120)$ to the values found in Problem 1 (in each program, the value of $U(0,20)$ generated by the program is shown). Remember that in the numerical program $r = i \cdot \Delta r$ and $t = j \cdot \Delta t$. Be sure to state what values you used for N and J.