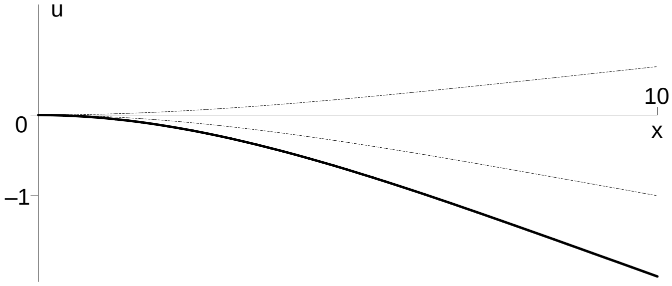
M344 - ADVANCED ENGINEERING MATHEMATICS

Lecture 21: Motion of a Deflected Beam

The deflection of a cantilever beam that is fixed at one end and free to move at the other can be shown to satisfy a *fourth-order* partial differential equation of the form

$$u_{tt} = -c^2 u_{xxxx},$$

where $u(x,t)$ is the deflection at the point x along the length of the beam, at time t. The positive constant c^2 is $\frac{EI}{\rho A}$, where A is the cross-sectional area of the beam, ρ is its density, E is the modulus of elasticity, and I is the moment of inertia.



The end conditions on the beam are modelled by assuming the following boundary conditions on the function u:

$$u(0,t) = u_x(0,t) = 0;$$

$$u_{xx}(L,t) = u_{xxx}(L,t) = 0, \quad \text{for all } t > 0.$$

Using the method of Separation of Variables, let $u(x,t) = X(x)T(t)$. Then

$$\frac{XT''}{c^2XT} = -\frac{c^2X''''T}{c^2XT} \Rightarrow$$

$$\frac{T''}{c^2T} = -\frac{X''''}{X} \equiv -\beta^4 \text{ for some constant } \beta^4.$$

This results in the two ordinary differential equations

$$X'''' - \beta^4 X = 0 \text{ and } T'' + c^2\beta^4 T = 0.$$

The solution of the second-order equation in T will have the form

$$T_n(t) = a_n \cos(c\beta_n^2 t) + b_n \sin(c\beta_n^2 t),$$

where the constants β_n^4 are the eigenvalues of the equation in X.

Solution of $X'''' - \beta^4 X = 0, \quad X(0) = X'(0) = X''(L) = X'''(L) = 0$

The characteristic polynomial for this fourth-order boundary-value problem is $r^4 - \beta^4 = 0$, and it has the four roots $r = \pm\beta, \pm\beta i$; therefore, the **general solution** of the differential equation can be written as

$$X(x) = A\cos(\beta x) + B\sin(\beta x) + C\cosh(\beta x) + D\sinh(\beta x),$$

and differentiating with respect to x,

$$X'(x) = -\beta A\sin(\beta x) + \beta B\cos(\beta x) + \beta C\sinh(\beta x) + \beta D\cosh(\beta x).$$

Using the boundary conditions at $x = 0$,

$$X(0) = A + C = 0 \Rightarrow C = -A;$$

and

$$X'(0) = \beta B + \beta D = \beta(B + D) = 0 \Rightarrow D = -B.$$

We can now write

$$X(x) = A(\cos(\beta x) - \cosh(\beta x)) + B(\sin(\beta x) - \sinh(\beta x))$$

$$X'(x) = \beta A(-\sin(\beta x) - \sinh(\beta x)) + \beta B(\cos(\beta x) - \cosh(\beta x))$$

$$X''(x) = \beta^2 A(-\cos(\beta x) - \cosh(\beta x)) + \beta^2 B(-\sin(\beta x) - \sinh(\beta x))$$

$$X'''(x) = \beta^3 A(\sin(\beta x) - \sinh(\beta x)) + \beta^3 B(-\cos(\beta x) - \cosh(\beta x))$$

The final two boundary conditions, at $x = L$, give

$$X''(L) = \beta^2 A(-\cos(\beta L) - \cosh(\beta L)) + \beta^2 B(-\sin(\beta L) - \sinh(\beta L)) = 0$$

$$X'''(L) = \beta^3 A(\sin(\beta L) - \sinh(\beta L)) + \beta^3 B(-\cos(\beta L) - \cosh(\beta L)) = 0$$

and these two equations can be written in matrix form as

$$\begin{pmatrix} \cos(\beta L) + \cosh(\beta L) & \sin(\beta L) + \sinh(\beta L) \\ -\sin(\beta L) + \sinh(\beta L) & \cos(\beta L) + \cosh(\beta L) \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \qquad (1)$$

This system of linear equations in A and B will have a *unique* solution $A = B = 0$ unless the determinant of the matrix is zero. This means that there will be non-zero solutions if, and only if,

$$(\cos(\beta L)+\cosh(\beta L))(\cos(\beta L)+\cosh(\beta L))+(\sin(\beta L)+\sinh(\beta L))(\sin(\beta L)-\sinh(\beta L))=0.$$

Multiplying out and using the two identities $(\sin(x))^2 + (\cos(x))^2 \equiv 1$ and $(\cosh(x))^2 - (\sinh(x))^2 \equiv 1$, results in the equation

$$(\cos(\beta L))^2 + 2\cosh(\beta L)\cos(\beta L) + (\cosh(\beta L))^2 + (\sin(\beta L))^2 - (\sinh(\beta L))^2$$

$$= 2 + 2\cosh(\beta L)\cos(\beta L) = 0. \quad (Checkthis!)$$

This means that the eigenvalues β_n must satisfy the condition

$$\cosh(\beta_n L)\cos(\beta_n L) = -1.$$

There will be an infinite sequence of positive solutions $L\beta_1 < L\beta_2 < \cdots$, tending to ∞, which can be found by using the following MAPLE instructions:

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for n from 1 to 15 do
fsolve(cosh(x)*cos(x)=-1.0, x=(n-1)*Pi..n*Pi); od;
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For n equal $1, 2,$ and 3 the solutions are $L\beta_1 \approx 1.8751, L\beta_2 \approx 4.69409, L\beta_3 \approx 7.8458$; and as $n \to \infty$, the solution $L\beta_n$ approaches the value $\frac{(2n-1)}{2}\pi$. The corresponding eigenfunctions $X_n(x)$ are

$$X_n(x) = A_n(\cos(\beta_n x) - \cosh(\beta_n x)) + B_n(\sin(\beta_n x) - \sinh(\beta_n x)),$$

and Equation (1) implies that B_n and A_n are related by

$$B_n = -\left(\frac{\cos(\beta_n L) + \cosh(\beta_n L)}{\sin(\beta_n L) + \sinh(\beta_n L)}\right) A_n.$$

Letting $A_n = 1$, we have a set of eigenfunctions of the form

$$\phi_n(x) = (\cos(\beta_n x) - \cosh(\beta_n x)) - \left(\frac{\cos(\beta_n L) + \cosh(\beta_n L)}{\sin(\beta_n L) + \sinh(\beta_n L)}\right)(\sin(\beta_n x) - \sinh(\beta_n x)). \tag{2}$$

The general solution of the partial differential equation can now be written in the form

$$u(x,t) = \sum_{n=1}^{\infty} \phi_n(x) T_n(t) = \sum_{n=1}^{\infty} \phi_n(x)(a_n \cos(c\beta_n^2 t) + b_n \sin(c\beta_n^2 t)).$$

Two initial conditions on $u(x,t)$ are needed to determine the coefficients a_n and b_n. These conditions will be given by specifying the initial deflection

$u(x,0) = f(x)$ and the initial velocity $u_t(x,0) = g(x)$ at each point x along the beam. In order to use these functions to obtain the coefficients a_n and b_n, it must first be shown that the functions $\phi_n(x)$ form an **orthogonal set** on the interval $[0,L]$.

Theorem 1 *The set of functions* $\{\phi_n(x)\}_1^\infty$ *defined in equation (2) is an orthogonal family on* $[0,L]$*; that is* $\int_0^L \phi_m(x)\phi_n(x)dx = 0$ *if* $m \neq n$*.*

Proof: We first use integration by parts to show that if u and v are any sufficiently differentiable functions on $0 \leq x \leq L$, then

$$\int_0^L u''''vdx = v(x)u'''(x)|_0^L - \int_0^L u'''v'dx$$

$$= v(x)u'''(x)|_0^L - \left(v'(x)u''(x)|_0^L - \int_0^L u''v''dx\right)$$

$\cdots$ and integrating by parts two more times $\cdots$

$$= v(x)u'''(x)|_0^L - v'(x)u''(x)|_0^L + v''(x)u'(x)|_0^L - v'''(x)u(x)|_0^L + \int_0^L uv''''dx,$$

If the right-hand term is subtracted from both sides,

$$\int_0^L u''''vdx - \int_0^L uv''''dx = (v(x)u'''(x) - v'(x)u''(x) + v''(x)u'(x) - v'''(x)u(x))|_0^L. \tag{3}$$

Now assume $u = \phi_m$ and $v = \phi_n$ where ϕ_m and ϕ_n are eigenfunctions of $X'''' - \beta^4 X = 0$ for two different eigenvalues β_n and β_m. Then $\phi_m'''' = \beta_m^4\phi_m$ and $\phi_n'''' = \beta_n^4\phi_n$, and equation (3) shows that

$$\int_0^L \phi_m''''\phi_n dx - \int_0^L \phi_m\phi_n'''' dx = \int_0^L \beta_m^4\phi_m\phi_n dx - \int_0^L \phi_m\beta_n^4\phi_n dx$$

$$= (\beta_m^4 - \beta_n^4)\int_0^L \phi_m\phi_n dx =$$

$$(\phi_n(x)\phi_m'''(x) - \phi_n'(x)\phi_m''(x) + \phi_n''(x)\phi_m'(x) - \phi_n'''(x)\phi_m(x))|_0^L$$

When the right-hand side of the equation is evaluated at the two endpoints $x = 0$ and $x = L$, the boundary conditions on ϕ_m and ϕ_n can be seen to imply that every product has one term equal to zero; therefore, $(\beta_m^4 - \beta_n^4)\int_0^L \phi_m\phi_n dx = 0$, and since $\beta_m \neq \beta_n$, this means that $\int_0^L \phi_m\phi_n dx = 0$ as required. ■

The values of the coefficients a_n and b_n can now be found by using the initial conditions. Since

$$u(x,0) \equiv f(x) = \sum_{n=1}^{\infty} \phi_n(x)(a_n \cos(0) + b_n \sin(0)) = \sum_{n=1}^{\infty} a_n \phi_n(x),$$

the a_n are the coefficients in an orthogonal series for $f(x)$; therefore,

$$a_n = \frac{\int_0^L f(x)\phi_n(x)dx}{\int_0^L (\phi_n(x))^2 dx}.$$

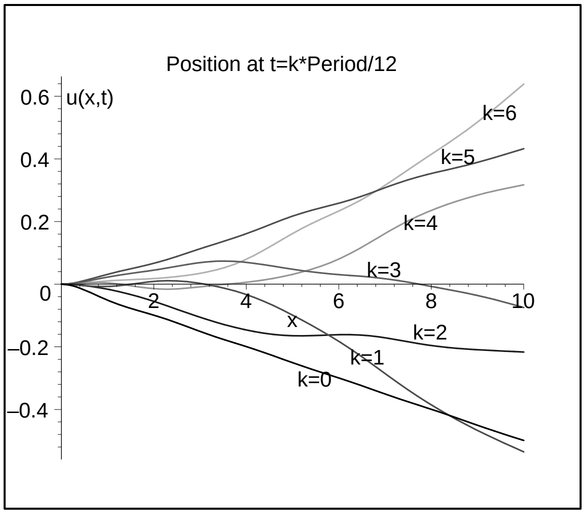
Similarly,

$$u_t(x,t) = \sum_{n=1}^{\infty} \phi_n(x) c\beta_n^2(-a_n \sin(c\beta_n^2 t) + b_n \cos(c\beta_n^2 t))$$

and

$$u_t(x,0) \equiv g(x) = \sum_{n=1}^{\infty} c\beta_n^2 b_n \phi_n(x) \Rightarrow b_n = \frac{1}{c\beta_n^2} \frac{\int_0^L g(x)\phi_n(x)dx}{\int_0^L (\phi_n(x))^2 dx}.$$

Example 1 *Assume that $c = 1$, and the initial displacement function is $f(x) = -0.05x$ on $0 \le x \le 10m$. The initial velocity is $g(x) \equiv 0$. Find the displacement $u(x,t)$ and plot the position of the beam at time increments of $\frac{1}{12}$ of the period of the function $T_1(t)$.*



In the figure above, the position of the beam at time $\frac{k}{12}\times$*period is labelled by* $k = 0, 1, \cdots, 6$. *The* **period** *of* $T_1(t) = a_1 \cos(c\beta_1^2) + b_1 \sin(c\beta_1^2)$*, in this particular example, is* $\frac{2\pi}{c\beta_1^2}$. *With* $c = 1$ *and* $L\beta_1 \approx 1.8751$*, the period is* $\frac{2\pi}{(0.18751)^2} \approx 178.72$ *and the curves labelled* $k = 0, 1, \cdots, 6$ *correspond to times* $t = 0, 14.9, 29.8, \cdots, 89.4$.

In the figure on page 1, the initial beam position was set equal to the first eigenfunction $\phi_1(x)$. In this case, a_1 is the only non-zero coefficient, and the beam will oscillate precisely with period $\frac{2\pi}{c\beta_1^2}$. For the straight-line initial position in the example, the oscillation is no longer exactly periodic because the periods of the functions $T_2, T_3, \cdots$ are not constant multiples of the period of T_1, and none of the coefficients a_n are zero for this straight-line initial position function.

Exercises:

1. * Determine the second eigenfunction $\phi_2(x)$ and plot a graph for $0 \leq x \leq 10$ (assume that the length of the beam is 10).
2. * Redo Example 1 and find $u(x,t)$, assuming that the initial displacement function is $f(x) = \phi_2(x)$ and the initial velocity function $g(x) \equiv 0$. Find the period of this function $u(x,t)$ and plot a graph showing the position of the beam at several different times during its period.