M344 - ADVANCED ENGINEERING MATHEMATICS

Lecture 21: Introduction to the Finite Element Method

The solution of Laplace's equation $u_{xx} + u_{yy} = 0$ on a rectangle can be found analytically in terms of an infinite series. If the region is made up of rectangular subregions then the numerical method, using finite differences to approximate the partial derivatives, can be used. There is a third method, the Method of Finite Elements, which can be used to obtain approximate solutions of partial differential equations on regions having arbitrary shapes. In this Lecture we will apply this method to an ordinary differential equation, in order to acquire an understanding of how the method works.

An ordinary differential equation (ODE) with enough initial conditions usually has a unique solution. If the conditions are not all given at the same value of the independent variable, then the problem becomes a boundary-value problem, and it is much harder to know if a unique solution, or any solution, exists.

We will consider the following example of a boundary-value problem:

$$y'' - 4y = t, \;\; 0 \le t \le 1, \;\; y(0) = y(1) = 0.$$

This is a linear second-order ode, and the general solution can be found by the characteristic polynomial method. If we assume $y(t) = e^{rt}$, and substitute y, y', and y'' into the equation, the characteristic polynomial is found to be $r^2 - 4 = 0$, with roots $r = \pm 2$. Therefore the homogeneous solution is $y_h(t) = c_1 e^{2t} + c_2 e^{-2t}$. Letting the particular solution be $y_p = At + B$, we find $y_p'' - 4y_p - t \equiv 0 = -4(At+B) - t$ and therefore $A = -\frac{1}{4}$ and $B = 0$. The general solution is $y(t) = y_h(t) + y_p(t)$, and using the two boundary conditions on y gives the unique solution $c_1 = \frac{1}{4(e^2 - e^{-2})}, c_2 = -c_1$. Therefore, the exact analytical solution of the boundary-value problem is

$$y(t) = \frac{1}{4}\left(\frac{e^{2t} - e^{-2t}}{e^2 - e^{-2}} - t\right).$$

To solve this problem by Finite Elements, we will write $y(t)$ as a linear combination of functions $\phi_k(t)$ which all satisfy the given boundary conditions. Then the approximate solution will have the form

$$y(t) \approx g(t) = \sum_{k=1}^{n} c_k \phi_k(t),$$

and the c_k are adjusted to make $g(t)$ approximate the solution of the differential equation as closely as possible. In this particular case, we will use only two functions $\phi_k(t)$, chosen to satisfy the two boundary conditions. Then the resulting function $g(t)$ will also satisfy $g(0) = g(1) = 0$, and will be a "good" approximation to the solution of the boundary-value problem if the constants c_k are chosen appropriately.

The two functions used in the approximation will be

$$\phi_1(t) = t(t-1) \text{ and } \phi_2(t) = t(t-\frac{1}{2})(t-1).$$

They were chosen to make the computation as easy as possible. Their only significant property is that they both satisfy the two the boundary conditions $\phi(0) = \phi(1) = 0$. To make $g(t)$ a good approximation to the differential equation, we will try to make the error function

$$E(t) = g''(t) - 4g(t) - t = \sum_{k=1}^{n} c_k(\phi_k''(t) - 4\phi_k(t)) - t$$

as small as possible on the interval $0 \leq t \leq 1$. Note that if $E(t)$ could be made *identically* zero on $[0,1]$, the function g would satisfy both the differential equation *and* the boundary conditions.

There are several ways to minimize the error function $E(t)$. If the functions $\{\phi_k\}_{k=1}^{\infty}$ formed an orthogonal set of functions on the interval $[0,1]$, then it would suffice to make $\int_0^1 E(t)\phi_k(t)dt = 0$ for all k . In the simple case studied here, we will arbitrarily set the function $E(t)$ equal to zero at two interior points of the interval, and see how close the resulting $g(t)$ comes to the exact solution $y(t)$ that was found above.

Using the two functions $\phi_1(t)$ and $\phi_2(t)$,

$$g(t) = A(t^2 - t) + B(t^3 - \frac{3}{2}t^2 + \frac{1}{2}t),$$

$$g'(t) = A(2t-1) + B(3t^2 - 3t + \frac{1}{2}),$$

$$g''(t) = A(2) + B(6t - 3),$$

where we have renamed c_1 and c_2 as A and B for convenience. Substituting g and g'' into $E(t)$,

$$E(t) = A(2 + 4t - 4t^2) + B(-4t^3 + 6t^2 + 4t - 3) - t.$$

Now set

$$E(\frac{1}{3}) = A(2 + \frac{4}{3} - \frac{4}{9}) + B(-\frac{4}{27} + \frac{6}{9} + \frac{4}{3} - 3) - \frac{1}{3} = 0$$

and

$$E(\frac{2}{3}) = A(2 + \frac{8}{3} - \frac{16}{9}) + B(-\frac{32}{27} + \frac{24}{9} + \frac{8}{3} - 3) - \frac{2}{3} = 0$$

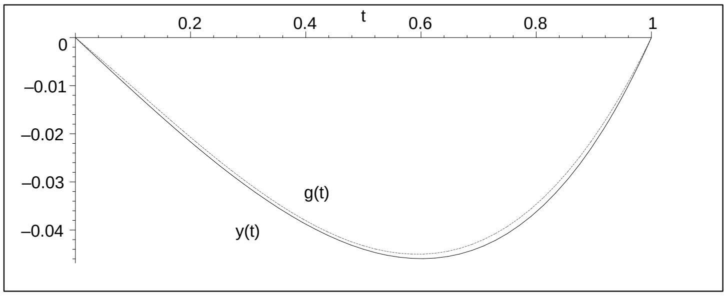
and solve the two resulting equations

$$\frac{26}{9}A - \frac{31}{27}B = \frac{1}{3}$$
$$\frac{26}{9}A + \frac{31}{27}B = \frac{2}{3}.$$

The solution is $A = \frac{9}{52}, B = \frac{9}{62}$, and the approximating function is

$$g(t) = \frac{9}{52}(t^2 - t) + \frac{9}{62}(t^3 - \frac{3}{2}t^2 + \frac{1}{2}t).$$

The figure below shows a plot of the solution $y(t)$ together with a plot of $g(t)$ (dotted). The agreement is surprisingly good considering the fact that only two functions ϕ_k were used.



When using the Finite Element Method on partial differential equations in two independent variables, the functions $\phi_k(x, y)$ are usually assumed to be piecewise linear functions of x and y and the subregions of the domain are triangles. It is then possible to write an error function $E(x, y)$ which, when minimized over the region in the (x, y)-plane where the p.d.e. is to be satisfied, gives a piecewise linear approximation to the solution. With a computer it is possible to break up the region of interest into larger and larger numbers of triangles, until the solution is exact to the required number of decimal places.

Practice Problems:

1. Redo the boundary-value problem

$$y'' - 4y = t, \;\; 0 \leq t \leq 1, \;\; y(0) = y(1) = 0$$

using the three functions:

$$\phi_1(t) = t(t-1), \quad \phi_2(t) = t(t-\frac{1}{2})(t-1), \quad \phi_3(t) = t(t-\frac{1}{3})(t-\frac{2}{3})(t-1).$$

There will now be *three* constants in the approximation

$$g(t) = A\phi_1(t) + B\phi_2(t) + C\phi_3(t),$$

and you can require that $E(\frac{1}{4}) = E(\frac{1}{2}) = E(\frac{3}{4}) = 0$. Does this result in a "better" approximation to the exact solution? Explain.