M344 - ADVANCED ENGINEERING MATHEMATICS

Lecture 20: The Wave Equation on a Rectangle

We will assume that a flexible membrane is stretched on a rectangle, under tension, and is kept fastened around the boundary. If it is given an initial velocity and/or acceleration, its displacement $u(x, y, t)$ can be assumed to satisfy the wave equation in *two* space dimensions given by:

$$u_{tt} = c^2(u_{xx} + u_{yy}). \tag{1}$$

Note that the constant $c^2 = \frac{T}{\rho}$ where T is the tension (force) and ρ is the linear density. The rectangle $R = \{0 \le x \le L, 0 \le y \le H\}$, and the initial and boundary conditions are given by

- Boundary conditions:
  $u(x, 0, t) = u(x, H, t) = 0$ for $0 \le x \le L, t > 0$
  $u(0, y, t) = u(L, y, t) = 0$ for $0 \le y \le H, t > 0$
- Initial conditions:
  $u(x, y, 0) = \alpha(x, y)$ (initial displacement for (x, y) in R)
  $u_t(x, y, 0) = \beta(x, y)$ (initial velocity for (x, y) in R)

Solution by the Method of Separation of Variables

First, let $u(x, y, t) = h(t)\Phi(x, y)$, and then substitute it into equation (1):

$$\frac{h''\Phi}{c^2 h\Phi} = \frac{c^2(h\Phi_{xx} + h\Phi_{yy})}{c^2 h\Phi} \Rightarrow \frac{h''}{c^2 h} = \frac{\Phi_{xx} + \Phi_{yy}}{\Phi} = -\lambda.$$

This leads to the two differential equations

$$h''(t) + \lambda c^2 h(t) = 0, \qquad \Phi_{xx}(x, y) + \Phi_{yy}(x, y) + \lambda\Phi(x, y) = 0.$$

Once the eigenvalues λ have been determined, it will be seen that they are all positive, so that the function $h(t)$ will be of the form

$$h(t) = c_1 \cos(c\sqrt{\lambda}t) + c_2 \sin(c\sqrt{\lambda}t).$$

To solve the equation

$$\Phi_{xx}(x, y) + \Phi_{yy}(x, y) + \lambda\Phi(x, y) = 0, \tag{2}$$

note that we can obtain homogeneous boundary conditions on $\Phi(x, y)$ from the conditions on u, as follows:

$$u(x, 0, t) = h(t)\Phi(x, 0) = 0, \quad \text{for } t > 0 \Rightarrow \Phi(x, 0) = 0 \text{ for all } x \text{ in } [0, L],$$

$$u(x, H, t) = h(t)\Phi(x, H) = 0, \quad \text{for } t > 0 \Rightarrow \Phi(x, H) = 0 \text{ for all } x \text{ in } [0, L],$$

$$u(0, y, t) = h(t)\Phi(0, y) = 0, \quad \text{for } t > 0 \Rightarrow \Phi(0, y) = 0 \text{ for all } y \text{ in } [0, H],$$

$$u(L, y, t) = h(t)\Phi(L, y) = 0, \quad \text{for } t > 0 \Rightarrow \Phi(L, y) = 0 \text{ for all } y \text{ in } [0, H].$$

Now let $\Phi(x, y) = X(x)Y(y)$. Then substituting into equation (2),

$$\frac{X''Y + XY'' + \lambda XY}{XY} = 0 \Rightarrow \frac{X''}{X} = -\frac{Y''}{Y} - \lambda = -\mu,$$

where μ is a second constant to be determined. First solve the Sturm-Liouville problem $X'' + \mu X = 0$. From the conditions above on the function Φ, we can get boundary conditions on X.

$$\Phi(0, y) = X(0)Y(y) = 0 \text{ for } 0 \le y \le H \Rightarrow X(0) = 0,$$

and

$$\Phi(L, y) = X(L)Y(y) = 0 \text{ for } 0 \le y \le H \Rightarrow X(L) = 0.$$

This is the same Sturm-Liouville problem we have solved many times before, and therefore we know that $X_n(x) = \sin(\frac{n\pi x}{L})$, with eigenvalues $\mu_n = \frac{n^2\pi^2}{L^2}, n = 1, 2, \cdots$.

Next consider the equation $Y'' = (\mu - \lambda)Y$. The conditions on Φ give homogeneous boundary conditions $Y(0) = Y(H) = 0$ for this equation also. For each integer $n = 1, 2, \cdots$, we will be able to find a sequence of eigenvalues $\lambda_{n1}, \lambda_{n2}, \cdots$ for which

$$Y''_{nm} + (\lambda_{nm} - \frac{n^2\pi^2}{L^2})Y_{nm} = 0.$$

This equation has solutions $Y_{nm}(y) = \sin(\frac{m\pi y}{H})$ with corresponding eigenvalues $\lambda_{nm} - \frac{n^2\pi^2}{L^2} = \frac{m^2\pi^2}{H^2}$. Therefore, for any integers $n = 1, 2, \cdots$ and $m = 1, 2, \cdots$,

$$\Phi_{nm} = X_n(x)Y_{nm}(y) = \sin(\frac{n\pi x}{L})\sin(\frac{m\pi y}{H}),$$

and

$$h_{nm}(t) = c_1\cos(c\sqrt{\lambda_{nm}}t) + c_2\sin(c\sqrt{\lambda_{nm}}t),$$

where

$$\lambda_{nm} = \frac{m^2\pi^2}{H^2} + \frac{n^2\pi^2}{L^2}.$$

This allows us to write the general solution of equation (1) in the form of a *doubly infinite* Fourier Series:

$$u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{nm}\sin(\frac{n\pi x}{L})\sin(\frac{m\pi y}{H})\cos(c\sqrt{\lambda_{nm}}t)$$

$$+\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} B_{nm}\sin(\frac{n\pi x}{L})\sin(\frac{m\pi y}{H})\sin(c\sqrt{\lambda_{nm}}t).$$

To find the coefficients A_{nm} and B_{nm}, we use the initial conditions on u:

$$u(x,y,0) = \sum_{m=1}^{\infty}\left(\sum_{n=1}^{\infty} A_{nm}\sin(\frac{n\pi x}{L})\right)\sin(\frac{m\pi y}{H}) = \alpha(x,y)$$

implies that $\sum_{m=1}^{\infty} A_{nm}\sin(\frac{n\pi x}{L})$ is the mth coefficient in the Fourier Sine Series for the function $\alpha(x,y)$, over the interval $0 \le y \le H$. Therefore,

$$\sum_{n=1}^{\infty} A_{nm}\sin(\frac{n\pi x}{L}) = \frac{2}{H}\int_0^H \alpha(x,y)\sin(\frac{m\pi y}{H})dy.$$

Note that this is a function $g(x)$, of the variable x, and A_{nm} is the nth coefficient in the Fourier Sine Series for the function $g(x)$ on $0 \le x \le L$. This means that we can write

$$A_{nm} = \frac{2}{L}\int_0^L g(x)\sin(\frac{n\pi x}{L})dx = \frac{2}{L}\int_0^L\left(\frac{2}{H}\int_0^H \alpha(x,y)\sin(\frac{m\pi y}{H})dy\right)\sin(\frac{n\pi x}{L})dx;$$

that is, A_{nm} is given by the *double integral*

$$A_{nm} = \frac{4}{LH}\int_0^L\int_0^H \alpha(x,y)\sin(\frac{m\pi y}{H})\sin(\frac{n\pi x}{L})dydx.$$

To find a formula for the B_{nm}, differentiate u with respect to t:

$$u_t(x,y,t) = -\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{nm}\sin(\frac{n\pi x}{L})\sin(\frac{m\pi y}{H})c\sqrt{\lambda_{nm}}\sin(c\sqrt{\lambda_{nm}}t)$$

$$+\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} B_{nm}\sin(\frac{n\pi x}{L})\sin(\frac{m\pi y}{H})c\sqrt{\lambda_{nm}}\cos(c\sqrt{\lambda_{nm}}t).$$

At $t = 0$, using the initial velocity function $\beta(x,y)$,

$$u_t(x,y,0) = \sum_{m=1}^{\infty}\left(\sum_{n=1}^{\infty} B_{nm}c\sqrt{\lambda_{nm}}\sin(\frac{n\pi x}{L})\right)\sin(\frac{m\pi y}{H}) = \beta(x,y);$$

therefore,

$$\sum_{n=1}^{\infty} B_{nm} c\sqrt{\lambda_{nm}} \sin(\frac{n\pi x}{L}) = \frac{2}{H}\int_0^H \beta(x,y)\sin(\frac{m\pi y}{H})dy,$$

and

$$B_{nm} = \frac{4}{c\sqrt{\lambda_{nm}}LH}\int_0^L \int_0^H \beta(x,y)\sin(\frac{m\pi y}{H})\sin(\frac{n\pi x}{L})dydx.$$

The solution to our problem can now be written as

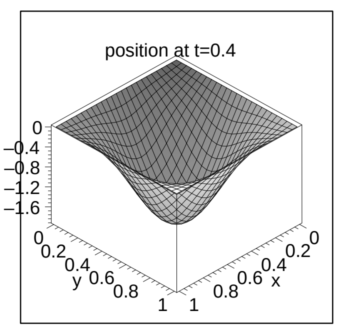
$$u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{nm}\sin(\frac{n\pi x}{L})\sin(\frac{m\pi y}{H})\cos(c\sqrt{\lambda_{nm}}t)$$

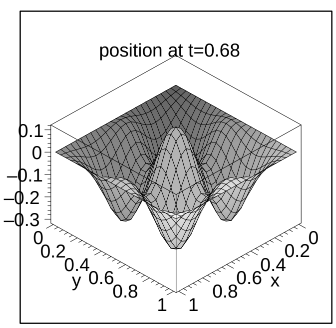
$$+\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} B_{nm}\sin(\frac{n\pi x}{L})\sin(\frac{m\pi y}{H})\sin(c\sqrt{\lambda_{nm}}t),$$

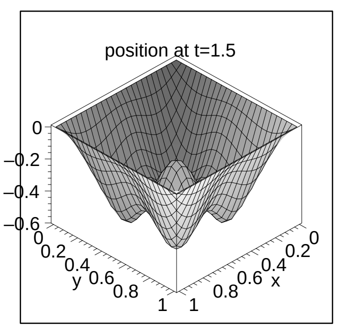
$$A_{nm} = \frac{4}{LH}\int_0^L \int_0^H \alpha(x,y)\sin(\frac{m\pi y}{H})\sin(\frac{n\pi x}{L})dydx,$$

$$B_{nm} = \frac{4}{c\sqrt{\lambda_{nm}}LH}\int_0^L \int_0^H \beta(x,y)\sin(\frac{m\pi y}{H})\sin(\frac{n\pi x}{L})dydx.$$

Example 1 *Assume a membrane is fastened on a rectangular frame which is one meter on each side. The membrane is initially at equilibrium, and is given an initial acceleration in the center. The initial position function is* $\alpha(x,y) \equiv 0$, *and the initial acceleration function is* $\beta(x,y) = -5.0exp(-(x-0.5)^2 - (y-0.5)^2)$ *for* $0 \le x \le 1, 0 \le y \le 1$. *Assume* $c = 1$.







Three views of the vibrating membrane, at times $t = 0.4, 0.68,$ *and* 1.5, *are pictured above, and a MAPLE program for solving this problem is shown below.*

Program to solve the Wave Equation on a Rectangle

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#Define the initial functions:
a:=(x,y)-> 0:  # initial position of the membrane
b:=(x,y)-> -5.0*exp(-((x-0.5)∧ 2 +(y-0.5)∧ 2)): # initial velocity
c:=1.0:  H:=1.0:  L:=1.0:  P:=evalf(Pi,10):  N:=6:  M:=6:
for n from 1 to N do
for m from 1 to M do lam[n,m]:=(n*P/L)∧ 2 + (m*P/H)∧ 2;
F[m]:=x->(2.0/H)*int(a(x,y)*sin(m*P*y/H),y=0..H);
G[m]:=x->(2.0/H)*int(b(x,y)*sin(m*P*y/H),y=0..H);
A[n,m]:=(2.0/L)*int(F[m](x)*sin(n*P*x/L),x=0..L);
B[n,m]:=(2.0/(L*c*sqrt(lam[n,m])))*int(G[m](x)*sin(n*P*x/L),x=0..L);
od; od:
u:=(x,y,t)->
sum(sum(A[j,k]*sin(j*P*x/L)*sin(k*P*y/H)*cos(c*sqrt(lam[j,k])*t),j=1..N),k=1..M)
+
sum(sum(B[s,r]*sin(s*P*x/L)*sin(r*P*y/H)*sin(c*sqrt(lam[s,r])*t),s=1..N),r=1..M);
with(plots):  plot3d(u(x,y,0),x=0..1.0,y=0..1.0,axes=BOXED);
PL:=animate3d(u(x,y,t),x=0..1.0,y=0..1.0,t=0..5.0,axes=BOXED,frames=100):
display(PL,insequence=true); ...  # view solution at t=0, 0.1, 0.2,...
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