M344 - ADVANCED ENGINEERING MATHEMATICS

Lecture 2: Review of Taylor Series

Def 1 *A function $f(t)$ is said to be* **analytic** *at $t = t_0$ if it is infinitely differentiable (i.e. all of its derivatives $f'(t_0), f''(t_0), \cdots$ exist at $t = t_0$) and the Taylor Series*

$$f(t) = f(t_0) + f'(t_0)(t - t_0) + \frac{f''(t_0)}{2!}(t - t_0)^2 + \frac{f'''(t_0)}{3!}(t - t_0)^3 + \cdots \qquad (1)$$

converges to $f(t)$ in some interval $(t_0 - \epsilon, t_0 + \epsilon)$ around t_0.

Note: Polynomials $p(t)$, and the functions $e^t, \sin(t)$, and $\cos(t)$ are analytic at every value of t. Rational functions $p(t)/q(t)$, where p and q are polynomials, are analytic at any t where $q(t) \neq 0$. The trig function $\tan(t)$, for example, is *not* analytic at odd multiples of $\frac{\pi}{2}$.

The Taylor Series for a function $f(t)$ is found by assuming it is an infinite-degree polynomial that has the same value as f at $t = t_0$, the same first derivative, the same second derivative, and so on. This allows one to show that the coefficients in equation (1) are $f^{(k)}(t_0)/k!$. In the special case when $t_0 = 0$, the Taylor Series is called a **Maclaurin Series**.

Def 2 *The* **factorial function** *$n!$ is defined, for positive integers n, by*

$$n! = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot n;$$

that is, $n!$ is the product of the first n non-zero integers. The value of $0!$ is defined to be one. The reason for this will be seen when we study the Γ function.

Example 1 *Find the Maclaurin series for $f(t) = \ln(1 + t)$.*

The series will have the form

$$f(t) = \ln(1 + t) = f(0) + f'(0)t + \frac{f''(0)}{2!}t^2 + \cdots .$$

To find the coefficients:

$$f(t) = \ln(1 + t) \to f(0) = \ln(1) = 0$$

$$f'(t) = \frac{1}{1 + t} = (1 + t)^{-1} \to f'(0) = 1$$

$$f''(t) = -(1 + t)^{-2} \to f''(0) = -1$$

$$f'''(t) = 2!(1 + t)^{-3} \to f'''(0) = 2!$$

and in general

$$f^{(n)}(t) = (-1)^{n+1}(n-1)!(1+t)^{-n} \rightarrow f^{(n)}(0) = (-1)^{n+1}(n-1)!$$

Therefore, $f(t) = 0 + t - \frac{1}{2!}t^2 + \frac{2!}{3!}t^3 - \frac{3!}{4!}t^4 + \cdots$ *and simplifying,*

$$f(t) = \ln(1+t) = t - \frac{1}{2}t^2 + \frac{1}{3}t^3 - \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} t^n.$$

Make sure that you understand this notation for the infinite sum.

There are certain series which come up often enough so that you should know what they look like. You should be able to recognize the following MacLaurin series:

- $\frac{1}{1-t} = 1 + t + t^2 + t^3 + \cdots$ (this is a geometric series)
- $e^t = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \cdots$
- $\sin(t) = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \cdots$
- $\cos(t) = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \cdots$

Euler's Formula, $e^{\theta\mathbf{i}} = \cos(\theta) + \sin(\theta)\mathbf{i}$, can now be justified by assuming that Taylor Series hold for complex numbers as well as real numbers. Writing out the series for $e^{\theta\mathbf{i}}$ by replacing t by $\theta\mathbf{i}$ in the above formula, we have

$$e^{\theta\mathbf{i}} = 1 + \theta\mathbf{i} + (\theta\mathbf{i})^2/2! + (\theta\mathbf{i})^3/3! + (\theta\mathbf{i})^4/4! + \cdots .$$

Now collect the real terms and the imaginary terms, and use the fact that $\mathbf{i}^2 = -1, \ \mathbf{i}^3 = -\mathbf{i}, \ \mathbf{i}^4 = 1, \ \cdots$ to write:

$$e^{\theta\mathbf{i}} = \left(1 - \theta^2/2! + \theta^4/4! + \cdots\right) + \left(\theta - \theta^3/3! + \theta^5/5! - \cdots\right)\mathbf{i}$$

$$\equiv \cos(\theta) + \sin(\theta)\mathbf{i}.$$

Convergence of Power Series

It can be shown that any convergent power series $\sum_{n=0}^{\infty} a_n(t-t_0)^n$ converges in some symmetric interval (t_0-R, t_0+R), called its **interval of convergence**. The real number R is called the **radius of convergence**. There are many tests for convergence of series, but the ones we will use most often are the **Ratio Test** and the **Alternating Series Test**.

Alternating Series Test:

If the terms of a series alternate in sign, for example $S = A_0 - A_1 + A_2 - A_3 + \cdots$, where the A_n are all positive, then the series converges if $A_{n+1} < A_n$ for all n, and $lim_{n\rightarrow\infty} A_n = 0$.

Ratio Test:

If $f(t) = \sum_{n=0}^{\infty} a_n(t-t_0)^n$ and $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = L$, then the radius of convergence of $f(t)$ about $t = t_0$ is $R = 1/L$. If $L = 0$, the series converges for all $t \in (-\infty, \infty)$; and if $L = \infty$, the series converges only at the point t_0.

Example 2 *Test the Maclaurin series* $ln(1+t) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} t^n$ *for convergence.*

By the Ratio Test,

$$lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(-1)^{n+2}/(n+1)}{(-1)^{n+1}/n}\right| = \lim_{n\to\infty} \left|\frac{-n}{n+1}\right| = 1 = L.$$

Therefore, $R = 1/L = 1$ *and the series converges to* $\ln(1+t)$ *on* $(-1,1)$. *Note that convergence at the two endpoints has to be checked by some other method.*

Operations on Taylor Series

The following properties are true for power series expanded about any $t = t_0$, but we will assume for convenience that we are working with Maclaurin series.

If two series $f(t) = \sum_{n=0}^{\infty} a_n t^n$ and $g(t) = \sum_{n=0}^{\infty} b_n t^n$, are both convergent in a common interval $(-R, R)$, then

$$f(t) \pm g(t) = \sum_{n=0}^{\infty} (a_n \pm b_n) t^n \text{ on } (-R, R)$$

$$f(t) \cdot g(t) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k b_{n-k} \right) t^n \text{ on } (-R, R)$$

Furthermore,

$$f'(t) = \frac{d}{dt} f(t) = \sum_{n=0}^{\infty} n a_n t^{n-1}$$

$$\int f(s) ds = \sum_{n=0}^{\infty} a_n \frac{t^{n+1}}{n+1} + C$$

both hold on $(-R, R)$.

Example 3 *Products of series can be computed in different ways. To obtain a Maclaurin series for* $f(t) = \frac{e^t}{1-t}$ *you could simply compute the derivatives* $f'(0), f''(0), \cdots$. *After about two of these, it gets very messy. A better way would be to use the formula given above for multiplying series and write:*

$$f(t) = e^t \cdot \frac{1}{1-t} = \left(\sum_{n=0}^{\infty} \frac{1}{n!} t^n \right) \cdot \left(\sum_{n=0}^{\infty} 1 \cdot t^n \right) \equiv \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{1}{k!}(1) \right) t^n;$$

therefore,

$$e^t \cdot \frac{1}{1-t} = \frac{1}{0!}t^0 + \left(\frac{1}{0!} + \frac{1}{1!}\right)t^1 + \left(\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!}\right)t^2 + \cdots$$

$$= 1 + 2t + \frac{5}{2}t^2 + \cdots .$$

To check the answer, the MAPLE instruction `series(exp(t)/(1-t),t=0,6)` *will produce the result*

$$1 + 2 * t + 5/2 * t^2 + 8/3 * t^3 + 65/24 * t^4 + 163/60 * t^5 + O(t^6),$$

which is the first 6 *terms in the Taylor Series for* $f(t)$*, expanded about* $t = 0$*. The term* $\mathrm{O}(t^6)$ *at the end is read "order of* t^6*". It implies that the error made in dropping the rest of the series is essentially equal to a constant times* t^6 *when* t *is close to zero.*

Example 4 *Find the Maclaurin series for* $\frac{1}{1+t}$ *by differentiating the series for* $f(t) = \ln(1+t)$*.*

$$\frac{1}{1+t} = \frac{d}{dt}(\ln(1+t)) = \frac{d}{dt}\left(\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} t^n\right) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} n t^{n-1}$$

$$= \sum_{n=1}^{\infty} (-1)^{n+1} t^{n-1} = 1 - t + t^2 - t^3 + \cdots .$$

This holds for all t *in the interval of convergence for the series; that is, for* $t \in (-1, 1)$*.*

This same series could have been found by substituting $s = -t$ into the series for $g(t) = \frac{1}{1-t} = 1 + t + t^2 + t^3 + \cdots$. This would result in the series $\frac{1}{1-(-t)} \equiv \frac{1}{1+t} = 1 + (-t) + (-t)^2 + (-t)^3 + \cdots = 1 - t + t^2 - t^3 + \cdots$, which is the series we found above. Any method that works will produce exactly the same Maclaurin series for a given function, so it always makes sense to use the simplest method available.

Approximating a numerical value of $f(t)$ by its series

To approximate the value of the series for an analytic function $f(t)$ at some t within its interval of convergence, we can use a **Taylor Polynomial of degree N**; that is, $f(t) \approx P_N(t) = f(t_0) + f'(t_0)(t-t_0) + \cdots + \frac{f^{(n)}(t_0)}{n!}(t-t_0)^n$.

A theorem, usually proved in Advanced Calculus, states that the error in $P_N(t)$ as an approximation to $f(t)$ is less in absolute value than $\max_\xi \left|\frac{f^{N+1}(\xi)(t-t_0)^{N+1}}{(N+1)!}\right|$, where the maximum value of the derivative is taken over values of ξ between t_0 and the value of t at which the approximation is made.

Example 5 *Use the Taylor polynomial of degree* 5 *for* $f(t) = \ln(1+t)$ *to obtain an approximation to* $\ln(1.5)$*. The exact value of* $\ln(1.5)$ *to seven significant digits, using the TI-89, is* 0.4054651*.... Compare this to*

$$ln(1 + 0.5) \approx 0.5 - (0.5)^2/2 + (0.5)^3/3 - (0.5)^4/4 + (0.5)^5/5 = 0.4072917.$$

It can be seen that the absolute error in the approximation is $|0.4054651 - 0.4072917| \approx 0.00183$*. From the error formula, this should be less than the maximum of* $|f^{(6)}(\xi)(0.5)^6/6!|$ *where* ξ *can be any value between* 0 *and* 0.5*. The formula for the sixth derivative of* f *is* $f^{(6)}(\xi) = -(5!)(1+\xi)^{-5}$ *and this takes its maximum absolute value of* $5!$ *at* $\xi = 0$*; therefore, the error formula states that the absolute error can be no greater than* $5!(0.5)^6/6! \approx 0.0026$*. We found the actual error to be smaller than this, but it can be seen that the error formula gives a value in the right range. The error formula can be very useful in finding a bound on the error made if the Taylor Polynomial is used to approximate the function over some interval of* t*.*

Practice Problems:

1. * Derive the Maclaurin series for $f(t) = e^t$ by evaluating the coefficients $f(0), f'(0), \cdots$. Write the series in summation form, and use the Ratio Test to show that it converges for all t.

2. Find the MacLaurin series for each of the following functions. Use the simplest possible method, and write the series in summation form.

   (a) $\sin(2t)$ Ans: $\sum_0^\infty (-1)^n (2t)^{2n+1}/(2n+1)!$

   (b) $\frac{1}{1+t^2}$ Ans: $\sum_0^\infty (-1)^n t^{2n}$

   (c) * $\frac{1}{(1-t)^2}$ Hint: $\frac{1}{(1-t)^2} = \frac{d}{dt}(\frac{1}{1-t})$.

3. * Find the first three non-zero terms in the series for $f(t) = 2\sin(t)\cos(t)$ by multiplying together the series for $\sin(t)$ and $\cos(t)$. Show that this is the same as the series in problem 2(a). Why?

4. Find the interval of convergence for

   (a) $\sum_{n=0}^\infty \frac{(t-2)^n}{3^n n}$ Ans: $(-1, 5)$

   (b) $\sum_{n=0}^\infty \frac{3^n t^n}{n!}$ Ans: $(-\infty, \infty)$

   (c) $\sum_{n=0}^\infty \frac{n!(t+1)^n}{2^n}$ Ans: $R = 0$, converges only at $t = -1$

5. * Write out the terms in $P_5(t)$, the Taylor polynomial of degree 5, about $t_0 = 0$, for $e^{-0.2t}$. Use MAPLE to graph both $e^{-0.2t}$ and $P_5(t)$ on the same set of axes for $0 \le t \le 5$. Find the absolute error if $P_5(t)$ is used to estimate the value of $e^{-0.2t}$ at $t = 3$. Show that this satisfies the bound given by the error formula.

6. * The Maclaurin series for $\tan(t)$ can be found by setting

$$\tan(t) \equiv a_0 + a_1 t + a_2 t^2 + \cdots \equiv \frac{\sin(t)}{\cos(t)};$$

so that multiplying both sides by $\cos(t)$, and replacing the sin and cos by their series,

$$(a_0 + a_1 t + a_2 t^2 + \cdots)(1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \cdots) \equiv t - \frac{t^3}{3!} + \frac{t^5}{5!} - \cdots .$$

Now multiply the series on the left (you will need at least 6 terms in the product - the first 3 are shown below):

$$a_0 + a_1 t + (a_2 - \frac{a_0}{2!})t^2 + \cdots \equiv t - \frac{t^3}{3!} + \frac{t^5}{5!} - \cdots .$$

A mathematical theorem says that these two expressions will be equal for all t for which they converge if, and only if, the coefficients of each power of t are the same on both sides. Use this to find the first 5 non-zero terms in the series for $\tan(t)$. Your answer can be checked by using the MAPLE instruction: `series(tan(t),t=0,10)`, which prints out the first 10 terms in the Maclaurin series for $\tan(t)$.