

## M344 - ADVANCED ENGINEERING MATHEMATICS

### Lecture 19: A Vibrating Circular Drumhead

Consider the problem of a vibrating drumhead. Its displacement at time  $t$  satisfies the wave equation  $u_{tt} = \alpha^2 \nabla^2(u)$ . If the drum is circular, we will want to use cylindrical coordinates  $(r, \theta, z)$ . The Laplacian of  $u(x, y, z)$  in rectangular coordinates,  $\nabla^2(u) = u_{xx} + u_{yy} + u_{zz}$ , can be converted to cylindrical coordinates (see the Appendix) and has the form

$$\nabla^2(u) = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} + u_{zz}.$$

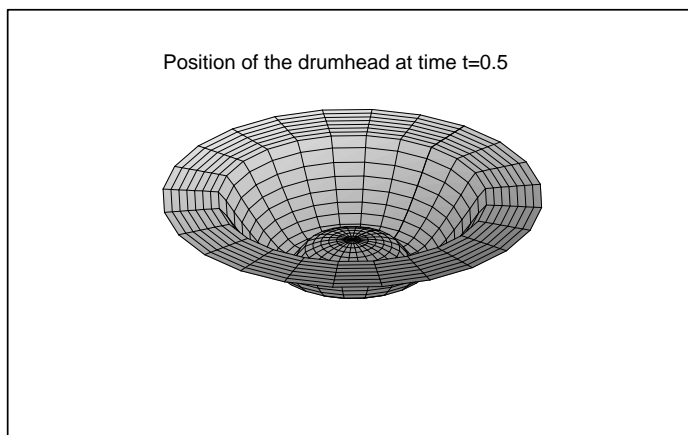


Figure 1: Drumhead initially struck in the center

If the drumhead is assumed to be a two-dimensional sheet, the term  $u_{zz}$  in the Laplacian can be considered to be zero. In order to make the partial differential equation 2-dimensional, it will also be assumed that the drumhead is initially displaced, and/or given an initial velocity, in such a way that  $u(r, \theta, 0)$  has the same value for all  $\theta$ , for a given value of  $r$ . This is called **radial symmetry**, and in this case  $u_{\theta\theta}$  is also identically zero. The wave equation that applies to the drumhead in this case has the form

$$u_{tt} = \alpha^2 \nabla^2(u) = \alpha^2 \left( u_{rr} + \frac{1}{r} u_r \right). \quad (1)$$

We will assume, without loss of generality, that  $\alpha^2 = 1$  and that the outer radius of the drumhead is  $r = 1$ . For all  $t > 0$ , the **boundary conditions** placed on the function  $u$  will then be given by  $u(1, t) = 0$  (that is, the drumhead is kept fastened

at its outer radius) and  $u(0, t)$  must be finite. Although the second condition seems a little unusual, when we find the series solution it will become clear why it is required. The initial displacement will be given by a function  $u(r, 0) = f(r)$ , which must be piecewise continuous on  $0 \leq r \leq 1$ . It represents the displacement along any radius from the center of the drumhead to the outer radius  $r = 1$ . Similarly, the initial velocity  $u_t(r, 0) = g(r)$  must also be a piecewise continuous function. The drumhead pictured in Figure 1 was initially given no displacement ( $f(r) \equiv 0$ ) but was hit in the center, giving it an initial velocity  $g(r) = -2$  for  $0 \leq r \leq 0.2$  and zero for  $0.2 < r \leq 1$ .

If we write  $u(r, t) = R(r)T(t)$ , and separate the variables in equation (1):

$$\frac{RT''}{RT} = \frac{\alpha^2(R''T + \frac{1}{r}R'T)}{RT} \Rightarrow \frac{T''}{T} = \frac{\alpha^2(R'' + \frac{1}{r}R')}{R} = -\lambda.$$

With  $\alpha^2 = 1$ , the equations in  $R$  and  $T$  are  $T'' + \lambda T = 0$  and  $R'' + \frac{1}{r}R' + \lambda R = 0$ . We do not have boundary conditions on the function  $T(t)$ , so our Sturm-Liouville equation is the equation in  $R$ . To see that this *is* a Sturm-Liouville equation, we write it in the form

$$rR'' + R' + \lambda rR = \frac{d}{dr}(rR') + \lambda rR = 0. \quad (2)$$

The boundary conditions  $u(1, t) = 0, u(0, t)$  finite, for all  $t > 0$  lead to boundary conditions  $R(1) = 0$  and  $R(0)$  finite for the Sturm-Liouville problem (2). Note that from what we learned about Sturm-Liouville problems, the eigenfunctions  $R_n(r)$  of equation (2) will be **orthogonal on the interval**  $[0, 1]$  with **weight function**  $\mathbf{w}(r) = r$ ; that is

$$\int_0^1 rR_n(r)R_m(r)dr = 0, \text{ for } m \neq n. \quad (3)$$

To find the general solution of the equation

$$rR'' + R' + \lambda rR = 0, \quad (4)$$

we will make a substitution that turns this equation into Bessel's equation of order zero. Remember that the Bessel equation of order zero has the form

$$t^2y'' + ty' + t^2y = 0, \quad (5)$$

and we know its general solution is

$$y(t) = c_1J_0(t) + c_2Y_0(t). \quad (6)$$

Make the change of independent variable  $r = \frac{1}{\sqrt{\lambda}}t$ , and let  $R(r) = y(t)$ . Then

$$y'(t) = \frac{d}{dt} (R(r(t))) = R'(r) \frac{dr}{dt} = \frac{1}{\sqrt{\lambda}} R'(r),$$

and

$$y''(t) = \frac{d}{dt} \left( \frac{1}{\sqrt{\lambda}} R'(r(t)) \right) = \frac{1}{\sqrt{\lambda}} R''(r) \frac{dr}{dt} = \frac{1}{\lambda} R''(r).$$

Now equation (5), with  $t = \sqrt{\lambda}r$ , becomes

$$\begin{aligned} (\lambda r^2) \left( \frac{1}{\lambda} R''(r) \right) + \sqrt{\lambda} r \left( \frac{1}{\sqrt{\lambda}} R'(r) \right) + \lambda r^2 R(r) &= 0 \\ r R''(r) + R'(r) + \lambda r R(r) &= 0, \end{aligned}$$

which is exactly the equation (4) we needed to solve. Since we already know its solution in terms of  $t$  is given by equation (6), we can now write the solution  $R(r)$  as

$$R(r) = y(t) = c_1 J_0(t) + c_2 Y_0(t) = c_1 J_0(\sqrt{\lambda}r) + c_2 Y_0(\sqrt{\lambda}r).$$

To satisfy the boundary condition requiring  $R(0) = c_1 J_0(0) + c_2 Y_0(0)$  to be finite, the constant  $c_2$  *must be zero*, since  $Y_0(0)$  is infinite. The second condition  $R(1) = c_1 J_0(\sqrt{\lambda}) = 0$  implies that  $\sqrt{\lambda}$  must be a zero of the Bessel function  $J_0$ ; that is,  $\sqrt{\lambda} = z_n$  where  $J_0(z_n) = 0$ . Therefore the eigenvalues are

$$\lambda = z_1^2, z_2^2, \dots, z_n^2, \dots,$$

where  $z_n$  is the  $n$ th zero of the Bessel function  $J_0$ , and we know from our study of Bessel functions in Lecture 6 that there exists an infinite set of zeros which tend to  $\infty$  as  $n \rightarrow \infty$ . The associated eigenfunctions are  $R_n(r) = c_n J_0(z_n r)$ .

To solve the corresponding equation for  $T_n$ ,  $T_n'' + \lambda_n T_n = T_n'' + z_n^2 T_n = 0$ , we use the characteristic polynomial and find that

$$T_n(t) = a_n \cos(z_n t) + b_n \sin(z_n t).$$

This leads to the series solution

$$u(r, t) = \sum_{n=1}^{\infty} R_n(r) T_n(t) = \sum_{n=1}^{\infty} J_0(z_n r) [a_n \cos(z_n t) + b_n \sin(z_n t)]. \quad (7)$$

To find the coefficients  $a_n$  and  $b_n$ , we use the initial conditions  $u(r, 0) = f(r)$  and  $u_t(r, 0) = g(r)$ . The first condition implies that

$$u(r, 0) = \sum_{n=1}^{\infty} a_n J_0(z_n r) \equiv f(r). \quad (8)$$

Because the functions  $J_0(z_n r)$  are solutions of a Sturm-Liouville problem, they are orthogonal and (8) is an orthogonal series for  $f(r)$  with coefficients given by

$$a_n = \frac{\int_0^1 r f(r) J_0(z_n r) dr}{\int_0^1 r (J_0(z_n r))^2 dr}.$$

Differentiating equation (7) with respect to  $t$ :

$$u_t(r, t) = \sum_{n=1}^{\infty} J_0(z_n r) [-z_n a_n \sin(z_n t) + z_n b_n \cos(z_n t)];$$

therefore,

$$u_t(r, 0) = \sum_{n=1}^{\infty} J_0(z_n r) z_n b_n \equiv g(r)$$

implies that  $z_n b_n$  are the coefficients in the orthogonal expansion of  $g(r)$  in terms of the eigenfunctions.

The **solution to the vibrating drumhead** can now be written as

$$u(r, t) = \sum_{n=1}^{\infty} J_0(z_n r) [a_n \cos(z_n t) + b_n \sin(z_n t)],$$

$$a_n = \frac{\int_0^1 r f(r) J_0(z_n r) dr}{\int_0^1 r (J_0(z_n r))^2 dr}, \quad b_n = \frac{1}{z_n} \frac{\int_0^1 r g(r) J_0(z_n r) dr}{\int_0^1 r (J_0(z_n r))^2 dr}.$$

**Example 1** Find the displacement of a circular drumhead which is initially hit by a drumstick so that  $u(r, 0) = f(r) \equiv 0$  and  $u_t(r, 0) = g(r) = -2$  if  $0 \leq r \leq 0.2$  and zero otherwise. The MAPLE program shown below can be used to compute the series solution to a problem of this form. Note that the  $n$ th zero  $z_n$  of  $J_0$  is obtained by using the instruction

$$\mathbf{z[n]} := \mathbf{fsolve(y(x) = 0, x = (n - 0.5) * P..n * P)};$$

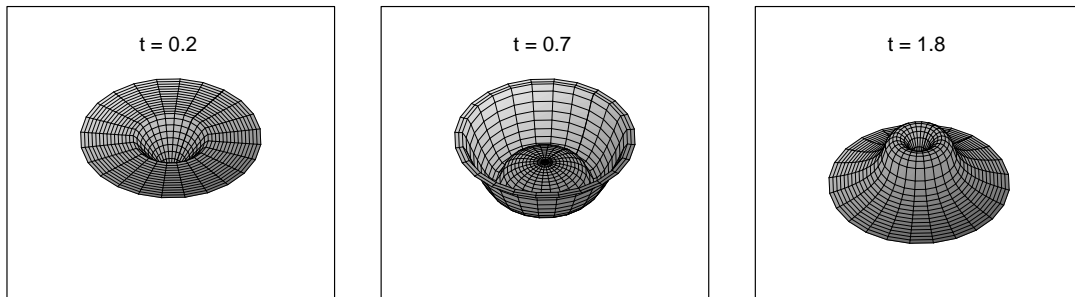
where  $y$  is the Bessel function  $J_0$  and  $P$  is a numerical value of  $\pi$ . The interval in which the  $n$ th zero lies was determined by using the fact from Lecture 6 that the function  $J_0(x)$  looks very much like a damped version of the function  $\cos(t - \frac{\pi}{4})$ , and its zeros  $a_n$  get closer and closer to  $(n - \frac{1}{4})\pi$  as  $n \rightarrow \infty$ .

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#Define the initial functions:
f:=r-> ... # initial position of the drumhead, 0 ≤ r ≤ 1
g:=r-> ... # initial velocity  $u_t(r,0)$ , 0 ≤ r ≤ 1
y:=x->BesselJ(0,x): # define  $y(x)$  to be the Bessel function  $J_0(x)$ 
P:=evalf(Pi,10): N:=30: #number of terms in the series for  $u$ 
for n from 1 to N do
z[n]:=fsolve(y(x)=0,x=(n-0.5)*P..n*P); # compute the nth zero of  $J_0$ 
a[n]:=int(r*f(r)*y(z[n]*r),r=0..1)/int(r*y(z[n]*r)*y(z[n]*r),r=0..1);
b[n]:=1/z[n]*int(r*g(r)*y(z[n]*r),r=0..1)/int(r*y(z[n]*r)*y(z[n]*r),r=0..1);
od:
u:=(r,t)->sum(y(z[m]*r)*(a[m]*cos(z[m]*t)+b[m]*sin(z[m]*t)),m=1..N);
with(plots): dt:=0.1:
frame:=j->cylinderplot([r,th,u(r,j*dt)],r=0..1,th=0..2.0*Pi,
orientation=[-144,21]);
display(frame(1)); display(frame(2)); ... # view solution at t=1,2,...

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The three figures below show the position of the drumhead at times  $t = 0.2, 0.7$ , and 1.8.



### Practice Problems:

- \* Use MAPLE to find the first 10 zeros of  $J_0(x)$  exact to 4 decimal places. Make a table comparing  $z_n$  to  $(n - \frac{1}{4})\pi$  for  $n$  from 1 to 10.
- \* Redo the problem in Example 1, where the initial functions are changed to  $f(r) = -0.3 + 0.3r$  if  $0 \leq r \leq 1.0$ , and the initial velocity  $g(r) \equiv 0$ . This simulates a drum that is pressed down in the center and let go at time  $t = 0$ . Compare the graphs you get at times  $t = 0.2, 0.7$ , and 1.8 to those found in Example 1.

## APPENDIX: Laplacian Converted to Cylindrical Coordinates

To convert  $\nabla^2(\mathbf{u}) = \mathbf{u}_{xx} + \mathbf{u}_{yy} + \mathbf{u}_{zz}$  to cylindrical coordinates  $(r, \theta, z)$  we first need the formulas for  $x, y$ , and  $z$  in terms of  $r, \theta$ , and  $z$ . In cylindrical coordinates:  $x = r \cos(\theta)$ ,  $y = r \sin(\theta)$ ,  $z = z$ .

The inverse formulas are

$$r = (x^2 + y^2)^{\frac{1}{2}}, \quad \theta = \tan^{-1}\left(\frac{y}{x}\right), \quad z = z.$$

To write  $u_{xx}$  and  $u_{yy}$  as functions of  $r, \theta$ , and  $z$ , we need the partial derivatives

$$\begin{aligned} \frac{\partial r}{\partial x} &= \frac{x}{r} = \cos(\theta), & \frac{\partial \theta}{\partial x} &= \frac{-y}{x^2 + y^2} = \frac{-\sin(\theta)}{r}, & \frac{\partial z}{\partial x} &= 0. \\ \frac{\partial r}{\partial y} &= \frac{y}{r} = \sin(\theta), & \frac{\partial \theta}{\partial y} &= \frac{x}{x^2 + y^2} = \frac{\cos(\theta)}{r}, & \frac{\partial z}{\partial y} &= 0. \end{aligned}$$

Let  $u(x, y, z) \equiv U(r, \theta, z)$ . Then, using the Chain Rule,

$$u_x = \frac{\partial u}{\partial x} = \frac{\partial U}{\partial r} \frac{\partial r}{\partial x} + \frac{\partial U}{\partial \theta} \frac{\partial \theta}{\partial x} + \frac{\partial U}{\partial z} \frac{\partial z}{\partial x} = U_r \cos(\theta) - U_\theta \frac{\sin(\theta)}{r}.$$

$$u_{xx} = \frac{\partial u_x}{\partial x} = \frac{\partial}{\partial r} \left( U_r \cos(\theta) - U_\theta \frac{\sin(\theta)}{r} \right) \frac{\partial r}{\partial x} + \frac{\partial}{\partial \theta} \left( U_r \cos(\theta) - U_\theta \frac{\sin(\theta)}{r} \right) \frac{\partial \theta}{\partial x}.$$

Check carefully that this results in the following 5 terms:

$$u_{xx} = U_{rr} \cos^2 \theta - 2U_{r\theta} \frac{\sin \theta \cos \theta}{r} + U_{\theta\theta} \frac{\sin^2 \theta}{r^2} + U_r \frac{\sin^2 \theta}{r} + 2U_\theta \frac{\sin \theta \cos \theta}{r^2}.$$

Using the same sequence of steps,

$$u_y = \frac{\partial u}{\partial y} = \frac{\partial U}{\partial r} \frac{\partial r}{\partial y} + \frac{\partial U}{\partial \theta} \frac{\partial \theta}{\partial y} = U_r \sin(\theta) + U_\theta \frac{\cos(\theta)}{r}$$

and therefore,

$$u_{yy} = U_{rr} \sin^2 \theta + 2U_{r\theta} \frac{\sin \theta \cos \theta}{r} + U_{\theta\theta} \frac{\cos^2 \theta}{r^2} + U_r \frac{\cos^2 \theta}{r} - 2U_\theta \frac{\sin \theta \cos \theta}{r^2}.$$

Adding the terms gives

$$u_{xx} + u_{yy} = U_{rr}(\sin^2 \theta + \cos^2 \theta) + U_r \left( \frac{\sin^2 \theta + \cos^2 \theta}{r} \right) + U_{\theta\theta} \left( \frac{\sin^2 \theta + \cos^2 \theta}{r^2} \right);$$

that is, the **Laplacian in cylindrical coordinates** is

$$\nabla^2(u) = u_{xx} + u_{yy} + u_{zz} \equiv U_{rr} + \frac{1}{r}U_r + \frac{1}{r^2}U_{\theta\theta} + U_{zz}.$$