Consider the problem of a vibrating drumhead. Its displacement at time $t$ satisfies the wave equation $u_{tt} = \alpha^2 \nabla^2 (u)$. If the drum is circular, we will want to use cylindrical coordinates $(r, \theta, z)$. The Laplacian of $u(x, y, z)$ in rectangular coordinates, $\nabla^2 (u) = u_{xx} + u_{yy} + u_{zz}$, can be converted to cylindrical coordinates (see the Appendix) and has the form

$$\nabla^2 (u) = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} + u_{zz}.$$ 

Figure 1: Drumhead initially struck in the center

If the drumhead is assumed to be a two-dimensional sheet, the term $u_{zz}$ in the Laplacian can be considered to be zero. In order to make the partial differential equation 2-dimensional, it will also be assumed that the drumhead is initially displaced, and/or given an initial velocity, in such a way that $u(r, \theta, 0)$ has the same value for all $\theta$, for a given value of $r$. This is called radial symmetry, and in this case $u_{\theta\theta}$ is also identically zero. The wave equation that applies to the drumhead in this case has the form

$$u_{tt} = \alpha^2 \nabla^2 (u) = \alpha^2 (u_{rr} + \frac{1}{r} u_r). \quad (1)$$

We will assume, without loss of generality, that $\alpha^2 = 1$ and that the outer radius of the drumhead is $r = 1$. For all $t > 0$, the boundary conditions placed on the function $u$ will then be given by $u(1, t) = 0$ (that is, the drumhead is kept fastened
at its outer radius) and \( u(0, t) \) must be finite. Although the second condition seems a little unusual, when we find the series solution it will become clear why it is required. The initial displacement will be given by a function \( u(r, 0) = f(r) \), which must be piecewise continuous on \( 0 \leq r \leq 1 \). It represents the displacement along any radius from the center of the drumhead to the outer radius \( r = 1 \).

Similarly, the initial velocity \( u_t(r, 0) = g(r) \) must also be a piecewise continuous function. The drumhead pictured in Figure 1 was initially given no displacement \((f(r) \equiv 0)\) but was hit in the center, giving it an initial velocity \( g(r) = -2 \) for \( 0 \leq r \leq 0.2 \) and zero for \( 0.2 < r \leq 1 \).

If we write \( u(r, t) = R(r)T(t) \), and separate the variables in equation (1):

\[
\frac{R''T}{RT} = \frac{\alpha^2(R''T + \frac{1}{r}R'T)}{R} \Rightarrow \frac{T''}{T} = \frac{\alpha^2(R'' + \frac{1}{r}R')}{R} = -\lambda.
\]

With \( \alpha^2 = 1 \), the equations in \( R \) and \( T \) are \( T'' + \lambda T = 0 \) and \( R'' + \frac{1}{r}R' + \lambda R = 0 \). We do not have boundary conditions on the function \( T(t) \), so our Sturm-Liouville equation is the equation in \( R \). To see that this is a Sturm-Liouville equation, we write it in the form

\[
rR'' + R' + \lambda rR = \frac{d}{dr} (rR') + \lambda rR = 0. \quad (2)
\]

The boundary conditions \( u(1, t) = 0, u(0, t) \) finite, for all \( t > 0 \) lead to boundary conditions \( R(1) = 0 \) and \( R(0) \) finite for the Sturm-Liouville problem (2). Note that from what we learned about Sturm-Liouville problems, the eigenfunctions \( R_n(r) \) of equation (2) will be orthogonal on the interval \([0, 1]\) with weight function \( w(r) = r \); that is

\[
\int_0^1 rR_n(r)R_m(r)dr = 0, \quad \text{for} \ m \neq n. \quad (3)
\]

To find the general solution of the equation

\[
rR'' + R' + \lambda rR = 0, \quad (4)
\]

we will make a substitution that turns this equation into Bessel’s equation of order zero. Remember that the Bessel equation of order zero has the form

\[
t^2y'' + ty' + t^2y = 0, \quad (5)
\]

and we know its general solution is

\[
y(t) = c_1 J_0(t) + c_2 Y_0(t). \quad (6)
\]
Make the change of independent variable $r = \frac{1}{\sqrt{\lambda}} t$, and let $R(r) = y(t)$. Then

$$y'(t) = \frac{d}{dt} (R(r(t))) = R'(r) \frac{dr}{dt} = \frac{1}{\sqrt{\lambda}} R'(r),$$

and

$$y''(t) = \frac{d}{dt} \left( \frac{1}{\sqrt{\lambda}} R'(r(t)) \right) = \frac{1}{\sqrt{\lambda}} R''(r) \frac{dr}{dt} = \frac{1}{\lambda} R''(r).$$

Now equation (5), with $t = \sqrt{\lambda} r$, becomes

$$(\lambda r^2) \left( \frac{1}{\sqrt{\lambda}} R''(r) \right) + \sqrt{\lambda} r \left( \frac{1}{\sqrt{\lambda}} R'(r) \right) + \lambda r^2 R(r) = 0$$

$$r R''(r) + R'(r) + \lambda r R(r) = 0,$$

which is exactly the equation (4) we needed to solve. Since we already know its solution in terms of $t$ is given by equation (6), we can now write the solution $R(r)$ as

$$R(r) = y(t) = c_1 J_0(\sqrt{\lambda} r) + c_2 Y_0(\sqrt{\lambda} r).$$

To satisfy the boundary condition requiring $R(0) = c_1 J_0(0) + c_2 Y_0(0)$ to be finite, the constant $c_2$ must be zero, since $Y_0(0)$ is infinite. The second condition $R(1) = c_1 J_0(\sqrt{\lambda}) = 0$ implies that $\sqrt{\lambda}$ must be a zero of the Bessel function $J_0$; that is, $\sqrt{\lambda} = z_n$ where $J_0(z_n) = 0$. Therefore the eigenvalues are

$$\lambda = z_1^2, z_2^2, \ldots, z_n^2, \ldots,$$

where $z_n$ is the nth zero of the Bessel function $J_0$, and we know from our study of Bessel functions in Lecture 6 that there exists an infinite set of zeros which tend to $\infty$ as $n \to \infty$. The associated eigenfunctions are $R_n(r) = c_n J_0(z_n r)$.

To solve the corresponding equation for $T_n$, $T_n'' + \lambda_n T_n = T_n'' + z_n^2 T_n = 0$, we use the characteristic polynomial and find that

$$T_n(t) = a_n \cos(z_n t) + b_n \sin(z_n t).$$

This leads to the series solution

$$u(r, t) = \sum_{n=1}^{\infty} R_n(r) T_n(t) = \sum_{n=1}^{\infty} J_0(z_n r) [a_n \cos(z_n t) + b_n \sin(z_n t)].$$

(7)

To find the coefficients $a_n$ and $b_n$, we use the initial conditions $u(r, 0) = f(r)$ and $u_t(r, 0) = g(r)$. The first condition implies that

$$u(r, 0) = \sum_{n=1}^{\infty} a_n J_0(z_n r) \equiv f(r).$$

(8)
Because the functions $J_0(z_nr)$ are solutions of a Sturm-Liouville problem, they are orthogonal and (8) is an orthogonal series for $f(r)$ with coefficients given by

$$a_n = \frac{\int_0^1 rf(r)J_0(z_nr)dr}{\int_0^1 r(J_0(z_nr))^2dr}.$$  

Differentiating equation (7) with respect to $t$:

$$u_t(r, t) = \sum_{n=1}^{\infty} J_0(z_nr)[-z_n a_n \sin(z_n t) + z_n b_n \cos(z_n t)];$$

therefore,

$$u_t(r, 0) = \sum_{n=1}^{\infty} J_0(z_nr)z_nb_n \equiv g(r)$$

implies that $z_nb_n$ are the coefficients in the orthogonal expansion of $g(r)$ in terms of the eigenfunctions.

The solution to the vibrating drumhead can now be written as

$$u(r, t) = \sum_{n=1}^{\infty} J_0(z_nr)[a_n \cos(z_n t) + b_n \sin(z_n t)],$$

$$a_n = \frac{\int_0^1 rf(r)J_0(z_nr)dr}{\int_0^1 r(J_0(z_nr))^2dr}, \quad b_n = \frac{1}{z_n} \frac{\int_0^1 rg(r)J_0(z_nr)dr}{\int_0^1 r(J_0(z_nr))^2dr}.$$  

Example 1 Find the displacement of a circular drumhead which is initially hit by a drumstick so that $u(r, 0) = f(r) \equiv 0$ and $u_t(r, 0) = g(r) = -2$ if $0 \leq r \leq 0.2$ and zero otherwise. The MAPLE program shown below can be used to compute the series solution to a problem of this form. Note that the $n$th zero $z_n$ of $J_0$ is obtained by using the instruction

$$z[n] := \text{fsolve}(y(x) = 0, x = (n - 0.5) * P..n * P);$$

where $y$ is the Bessel function $J_0$ and $P$ is a numerical value of $\pi$. The interval in which the $n$th zero lies was determined by using the fact from Lecture 6 that the function $J_0(x)$ looks very much like a damped version of the function $\cos(t - \frac{\pi}{4})$, and its zeros $a_n$ get closer and closer to $(n - \frac{1}{4})\pi$ as $n \to \infty$.  


Define the initial functions:

\[ f(r) = r \text{ for } 0 \leq r \leq 1 \]

\[ g(r) = 0 \text{ for } 0 \leq r \leq 1 \]

\[ y(x) = J_0(x) \text{ to be the Bessel function } J_0(x) \]

\[ P := \text{evalf}(\pi, 10); \]

\[ N := 30; \]

\[ \text{for } n \text{ from 1 to } N \text{ do} \]

\[ z[n] := \text{fsolve}(y(x) = 0, x = (n-0.5)P..nP); \]

\[ a[n] := \text{int}(r*f(r)*y(z[n]*r), r=0..1)/\text{int}(r*y(z[n]*r)*y(z[n]*r), r=0..1); \]

\[ b[n] := (1/z[n])*\text{int}(r*g(r)*y(z[n]*r), r=0..1)/\text{int}(r*y(z[n]*r)*y(z[n]*r), r=0..1); \]

\[ \text{od;} \]

\[ u := (r,t) \mapsto \sum m y(z[m]*r)*(a[m]*\cos(z[m]*t)+b[m]*\sin(z[m]*t)), m=1..N; \]

with(plots):

\[ \text{frame} := j \mapsto \text{cylinderplot}([r,\theta,u(r,j*dt)], r=0..1, \theta=0..2\pi, \text{orientation}=[-144,21]); \]

\[ \text{display(frame(1))}; \]

The three figures below show the position of the drumhead at times \( t = 0.2, 0.7, \) and 1.8.

Practice Problems:

1. * Use MAPLE to find the first 10 zeros of \( J_0(x) \) exact to 4 decimal places. Make a table comparing \( z_n \) to \( (n - \frac{1}{4})\pi \) for \( n \) from 1 to 10.

2. * Redo the problem in Example 1, where the initial functions are changed to \( f(r) = -0.3 + 0.3r \) if \( 0 \leq r \leq 1.0 \), and the initial velocity \( g(r) \equiv 0 \). This simulates a drum that is pressed down in the center and let go at time \( t = 0 \). Compare the graphs you get at times \( t = 0.2, 0.7, \) and 1.8 to those found in Example 1.
APPENDIX: Laplacian Converted to Cylindrical Coordinates

To convert $\nabla^2(u) = u_{xx} + u_{yy} + u_{zz}$ to cylindrical coordinates $(r, \theta, z)$ we first need the formulas for $x, y,$ and $z$ in terms of $r, \theta,$ and $z$. In cylindrical coordinates:

\[ x = r \cos(\theta), \quad y = r \sin(\theta), \quad z = z. \]

The inverse formulas are

\[ r = (x^2 + y^2)^{\frac{1}{2}}, \quad \theta = \tan^{-1}\left(\frac{y}{x}\right), \quad z = z. \]

To write $u_{xx}$ and $u_{yy}$ as functions of $r, \theta,$ and $z$, we need the partial derivatives

\[ \frac{\partial r}{\partial x} = \frac{x}{r} = \cos(\theta), \quad \frac{\partial \theta}{\partial x} = \frac{-y}{x^2 + y^2} = -\frac{\sin(\theta)}{r}, \quad \frac{\partial z}{\partial x} = 0. \]

\[ \frac{\partial r}{\partial y} = \frac{y}{r} = \sin(\theta), \quad \frac{\partial \theta}{\partial y} = \frac{x}{x^2 + y^2} = \frac{\cos(\theta)}{r}, \quad \frac{\partial z}{\partial y} = 0. \]

Let $u(x, y, z) \equiv U(r, \theta, z)$. Then, using the Chain Rule,

\[ u_x = \frac{\partial u}{\partial x} = \frac{\partial U}{\partial r} \frac{\partial r}{\partial x} + \frac{\partial U}{\partial \theta} \frac{\partial \theta}{\partial x} + \frac{\partial U}{\partial z} \frac{\partial z}{\partial x} = U_r \cos(\theta) - \frac{U_\theta \sin(\theta)}{r}. \]

\[ u_{xx} = \frac{\partial u_x}{\partial x} = \frac{\partial}{\partial r} \left( U_r \cos(\theta) - \frac{U_\theta \sin(\theta)}{r} \right) \frac{\partial r}{\partial x} + \frac{\partial}{\partial \theta} \left( U_r \cos(\theta) - \frac{U_\theta \sin(\theta)}{r} \right) \frac{\partial \theta}{\partial x}. \]

Check carefully that this results in the following 5 terms:

\[ u_{xx} = U_{rr} \cos^2 \theta - 2U_r \theta \frac{\sin \theta \cos \theta}{r} + U_{\theta \theta} \frac{\sin^2 \theta}{r^2} + U_r \frac{\sin^2 \theta}{r} + 2U_\theta \frac{\sin \theta \cos \theta}{r^2}. \]

Using the same sequence of steps,

\[ u_y = \frac{\partial u}{\partial y} = \frac{\partial U}{\partial r} \frac{\partial r}{\partial y} + \frac{\partial U}{\partial \theta} \frac{\partial \theta}{\partial y} = U_r \sin(\theta) + \frac{U_\theta \cos(\theta)}{r} \]

and therefore,

\[ u_{yy} = U_{rr} \sin^2 \theta + 2U_r \theta \frac{\sin \theta \cos \theta}{r} + U_{\theta \theta} \frac{\cos^2 \theta}{r^2} + U_r \frac{\cos^2 \theta}{r} - 2U_\theta \frac{\sin \theta \cos \theta}{r^2}. \]

Adding the terms gives

\[ u_{xx} + u_{yy} = U_{rr} (\sin^2 \theta + \cos^2 \theta) + U_r \left( \frac{\sin^2 \theta + \cos^2 \theta}{r} \right) + U_{\theta \theta} \left( \frac{\sin^2 \theta + \cos^2 \theta}{r^2} \right); \]

that is, the Laplacian in cylindrical coordinates is

\[ \nabla^2(u) = u_{xx} + u_{yy} + u_{zz} \equiv U_{rr} + \frac{1}{r} U_r + \frac{1}{r^2} U_{\theta \theta} + U_{zz}. \]