

M344 - ADVANCED ENGINEERING MATHEMATICS

Lecture 18: Laplace's Equation, Analytic and Numerical Solution

Our example of an **elliptic** partial differential equation is Laplace's equation, also called the Diffusion Equation. If the heat equation is written in terms of 3 space variables x, y , and z , it has the form

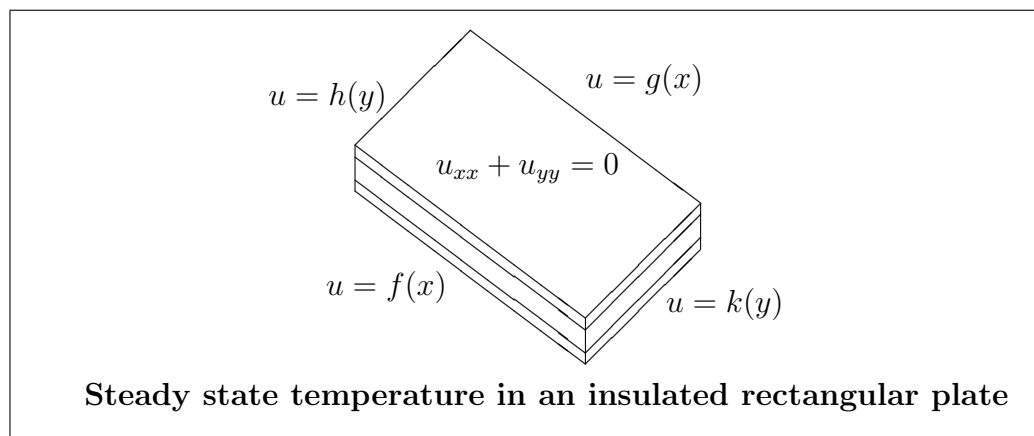
$$u_t = a^2(u_{xx} + u_{yy} + u_{zz}) = a^2\nabla^2u,$$

where ∇^2 is read "del squared", and is called the Laplacian operator. When the temperature u reaches steady state, that is, it stops changing with time, then $u_t = 0$ and we have the steady state heat equation $\nabla^2u = u_{xx} + u_{yy} + u_{zz} = 0$. This is Laplace's equation in three space variables. The equation we are going to solve is **Laplace's equation in two space variables**, which can be written as

$$\nabla^2u = u_{xx} + u_{yy} = 0. \quad (1)$$

It can be seen that equation (1) is an **elliptic** partial differential equation, according to our definition, by noting that $A = C = 1$ and $B = 0$ imply $B^2 - 4AC = -4 < 0$.

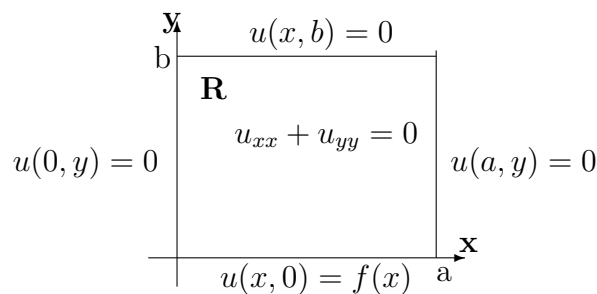
As an example of a physical situation where this equation arises, consider the temperature $u(x, y)$ in a rectangular metal plate which is insulated on the top and bottom, so that heat cannot flow in the z -direction. If the temperatures on all four edges of the rectangle are specified, then as $t \rightarrow \infty$, the temperature in the interior of the rectangular plate will approach the solution of equation (1).



Solution of Laplace's Equation by Separation of Variables

The method of separation of variables that we applied to the heat equation and the wave equation can also be used to solve equation (1), if it is assumed that three sides of the rectangle are held at temperature 0^0 . The temperature on the fourth side can be specified by an arbitrary piecewise continuous function. Since Laplace's equation is linear and homogeneous, we can find four different series solutions, each one satisfying a non-zero condition on a different side, and add them together to get a solution which satisfies arbitrary conditions around the entire boundary of the rectangle. Note that there is no initial condition on u , since we are looking for the **steady state** temperature inside the rectangle $R = \{0 \leq x \leq a, 0 \leq y \leq b\}$.

We will assume first that the boundary conditions are as shown in the figure below.



If $u(x, y) = X(x)Y(y)$ is substituted into equation (1), and the result is divided by XY ,

$$\frac{X''Y}{XY} + \frac{XY''}{XY} = 0 \Rightarrow \frac{X''}{X} = -\frac{Y''}{Y} = -\lambda.$$

The two ordinary differential equations in x and y are

$$X''(x) + \lambda X(x) = 0, \quad Y''(y) - \lambda Y(y) = 0.$$

The boundary conditions $u(0, y) = X(0)Y(y) = 0$ and $u(a, y) = X(a)Y(y) = 0$ for all y in the interval $[0, b]$ imply that $X(0) = X(a) = 0$. These boundary conditions give us the same Sturm-Liouville problem for $X'' + \lambda X = 0$ that we have solved twice before. The eigenvalues will be $\lambda_n = \frac{n^2\pi^2}{a^2}$ and the corresponding eigenfunctions are $X_n(x) = c_n \sin(\frac{n\pi x}{a})$. The equation for Y_n , with $\lambda_n = \frac{n^2\pi^2}{a^2}$, becomes $Y_n'' - \frac{n^2\pi^2}{a^2} Y_n = 0$, which has solution

$$Y_n(y) = a_n \cosh\left(\frac{n\pi y}{a}\right) + b_n \sinh\left(\frac{n\pi y}{a}\right).$$

The series solution, which will be called $u_1(x, y)$, can be written as

$$u_1(x, y) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{a}\right) \left[A_n \cosh\left(\frac{n\pi y}{a}\right) + B_n \sinh\left(\frac{n\pi y}{a}\right) \right],$$

and the coefficients A_n and B_n must be chosen to satisfy the remaining two conditions $u_1(x, b) = 0$, $u_1(x, 0) = f(x)$, for $0 \leq x \leq a$. The condition

$$u_1(x, 0) = f(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{a}\right)$$

implies that the A_n are the coefficients in the Fourier Sine Series for $f(x)$; therefore, $A_n = \frac{2}{a} \int_0^a f(x) \sin\left(\frac{n\pi x}{a}\right) dx$. The other condition implies that

$$u_1(x, b) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{a}\right) \left[A_n \cosh\left(\frac{n\pi b}{a}\right) + B_n \sinh\left(\frac{n\pi b}{a}\right) \right] \equiv 0,$$

and therefore, for each $n = 1, 2, \dots$, we must have $A_n \cosh\left(\frac{n\pi b}{a}\right) + B_n \sinh\left(\frac{n\pi b}{a}\right) = 0$. This means that

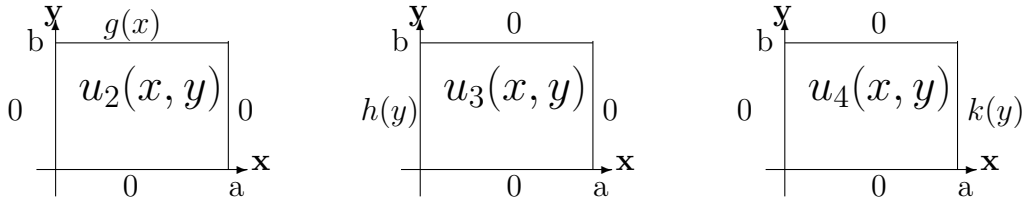
$$B_n = -A_n \frac{\cosh\left(\frac{n\pi b}{a}\right)}{\sinh\left(\frac{n\pi b}{a}\right)} = -A_n \coth\left(\frac{n\pi b}{a}\right),$$

and the solution u_1 can be written as

$$u_1(x, y) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{a}\right) A_{n1} \left[\cosh\left(\frac{n\pi y}{a}\right) - \coth\left(\frac{n\pi b}{a}\right) \sinh\left(\frac{n\pi y}{a}\right) \right],$$

$$A_{n1} = \frac{2}{a} \int_0^a f(x) \sin\left(\frac{n\pi x}{a}\right) dx.$$

The other three cases correspond to boundary conditions specified as shown below:



The corresponding solutions are:

$$\begin{aligned}
u_2(x, y) &= \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{a}\right) [B_{n2} \sinh\left(\frac{n\pi y}{a}\right)], \quad B_{n2} = \frac{\frac{2}{a} \int_0^a g(x) \sin\left(\frac{n\pi x}{a}\right) dx}{\sinh\left(\frac{n\pi b}{a}\right)} \\
u_3(x, y) &= \sum_{n=1}^{\infty} \sin\left(\frac{n\pi y}{b}\right) A_{n3} [\cosh\left(\frac{n\pi x}{b}\right) - \coth\left(\frac{n\pi a}{b}\right) \sinh\left(\frac{n\pi x}{b}\right)], \\
A_{n3} &= \frac{2}{b} \int_0^b h(y) \sin\left(\frac{n\pi y}{b}\right) dy \\
u_4(x, y) &= \sum_{n=1}^{\infty} \sin\left(\frac{n\pi y}{b}\right) [B_{n4} \sinh\left(\frac{n\pi x}{b}\right)], \quad B_{n4} = \frac{\frac{2}{b} \int_0^b k(y) \sin\left(\frac{n\pi y}{b}\right) dy}{\sinh\left(\frac{n\pi a}{b}\right)}
\end{aligned}$$

In the exercises you will be asked to derive the solution $u_4(x, y)$. In this case the Sturm-Liouville equation is the equation in y , rather than the equation in x .

Example 1 Consider the rectangle $R = \{0 \leq x \leq 15, 0 \leq y \leq 10\}$ with temperatures along the boundary given by $u(x, 0) = f(x) = 0.7x(15-x)$, $u(x, 10) = g(x) \equiv 0$, $u(0, y) = h(y) = 20 \sin\left(\frac{\pi y}{5}\right)$, and $u(15, y) = k(y) = y(10-y)$.

Using a MAPLE program to compute the coefficients, the series for $u(x, y) = u_1(x, y) + u_3(x, y) + u_4(x, y)$, with 20 terms in each series was plotted in Figure 1(a) as a three-dimensional surface above the rectangle R . A contour plot showing where the temperature has the values $-10^0, 0^0, 5^0, 10^0, 15^0, 20^0$ and 30^0 is also shown, in Figure 1(b). In this example, the boundary functions were chosen so that $u = 0$ at all four corners. This guaranteed a continuous solution everywhere inside the rectangle.

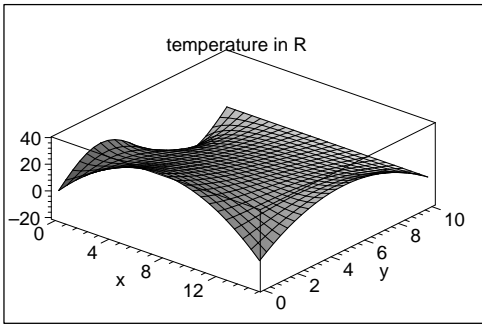


Figure 1: (a)

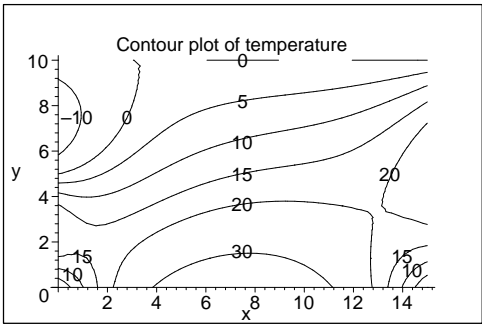


Figure 1: (b)

Numerical Solution of Laplace's Equation

If the rectangle R is partitioned along the x and y axes, by letting $\Delta x = a/N$ and $\Delta y = b/M$ for integers N and M , the central difference formula for u_{xx} and u_{yy} can be used to write the following difference approximation to equation (1):

$$\frac{u(x + \Delta x, y) - 2u(x, y) + u(x - \Delta x, y)}{(\Delta x)^2} + \frac{u(x, y + \Delta y) - 2u(x, y) + u(x, y - \Delta y)}{(\Delta y)^2} = 0.$$

If Δx and Δy can be chosen to be equal, then letting $\Delta x = \Delta y = h$, the equation can be multiplied by h^2 on both sides, resulting in

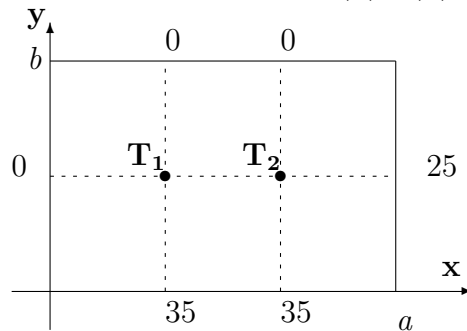
$$u(x + h, y) - 2u(x, y) + u(x - h, y) + u(x, y + h) - 2u(x, y) + u(x, y - h) = 0.$$

This can be solved for $u(x, y)$ in the form

$$4u(x, y) = u(x + h, y) + u(x - h, y) + u(x, y + h) + u(x, y - h). \quad (2)$$

Note that this says that the temperature at each grid point in the interior of R is the average of the temperatures at the four nearest grid points. If all of the boundary values are given, this produces a system of linear equations for the unknown temperatures in the grid. The number of equations in the linear system is equal to the number of interior grid points.

Example 2 We will numerically approximate the temperatures that were calculated by the series solution in Example 1. With $a = 15$ and $b = 10$, we can let $\Delta x = \Delta y = h = 5$, and use the grid shown below. There are only two unknown temperatures to be computed, labelled T_1 and T_2 . They should be approximations to $u(5, 5)$ and $u(10, 5)$, respectively. The boundary temperatures were obtained from the formulas for $f(x)$, $g(x)$, $h(y)$, and $k(y)$ in Example 1.



Using equation (2), the two linear equations for T_1 and T_2 are:

$$4T_1 = 0 + T_2 + 35 + 0, \quad 4T_2 = 0 + 25 + 35 + T_1$$

Written in the form

$$\begin{aligned} 4T_1 - T_2 &= 35 \\ -T_1 + 4T_2 &= 60 \end{aligned}$$

the equations can be solved to give $T_1 = 13.3333$ and $T_2 = 18.3333$. These compare to the values $T_1 = u(5, 5) \approx 12.06$ and $T_2 = u(10, 5) \approx 16.34$ obtained from the series solution in Example 1.

It is clear that a much better numerical approximation would result if the step size h is decreased. If we take $h = 2.5$, which is one half of the original h , the number of unknown temperatures inside the rectangle increases to 15 (Check it!). Similarly, the number of linear equations in the system increases to 15. Computer methods for solving large systems of linear equations have been around for a long time, and they are very easy to apply. This is a topic that is covered in both Linear Algebra and Numerical Analysis courses.

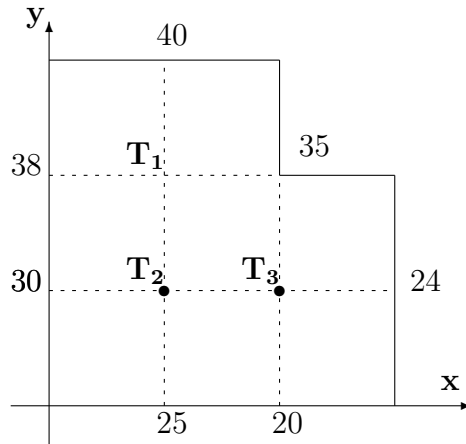
Practice Problems:

1. * Derive the series solution for $u(x, y)$ where u satisfies Laplace's equation inside the rectangle R , and the boundary conditions are:

$$u(x, 0) = u(x, b) = 0, \quad 0 \leq x \leq a; \quad u(0, y) = 0, \quad u(a, y) \equiv k(y), \quad 0 \leq y \leq b.$$

Your answer should look like $u_4(x, y)$ on page 4.

2. * Use the numerical method for solving Laplace's equation to find approximations to T_1, T_2 , and T_3 in the L-shaped region shown here. The boundary temperatures are all given in the diagram.



Note: The method of separation of variables does not work for a non-rectangular region of this type.

3. **Extra Credit:** Set up and solve the system of linear equations to find the numerical solution to Example 2, using $h = 2.5$. The corresponding 15 interior temperatures given by the series in Example 1 (with 20 terms) are:

x	2.5	5.0	7.5	10.0	12.5
$y = 7.5$	-1.0737	4.7192	7.0714	8.6165	11.176
$y = 5.0$	6.8162	12.059	15.055	16.337	18.086
$y = 2.5$	16.458	21.849	25.008	24.026	20.392

Compare your numerical solution to these values. Is the agreement better than it was with $h = 5$? Explain.

Hint: The 15×15 matrix you need can be set up as a 3×3 block matrix, where the blocks consist of 3 different 5×5 matrices. One of these is the zero matrix, another is $-I$, where I is the 5×5 identity matrix, and the third has a special form. To see how to set up a block matrix in MAPLE, execute the instruction `?blockmatrix`.

4. * If Δx and Δy are halved again, so that $h = 1.25$, how many unknown temperatures must be computed in the numerical solution in Example 2? Determine what the matrix of coefficients will look like for this system. Give a complete description of it as a block matrix.