

M344 - ADVANCED ENGINEERING MATHEMATICS

Lecture 14: Other Boundary Conditions for the Heat Equation

We have seen that the one-dimensional heat equation can be used to model the temperature in a rod which is insulated around its sides. The first case we solved assumed that both ends of the rod were held at 0^0 for all time $t > 0$. Other boundary conditions can be modelled, and we will treat two different cases.

Heat Equation with both ends insulated

If the ends of the rod are *insulated*, the boundary conditions on the partial differential equation $u_t = \alpha^2 u_{xx}$ must be changed to $u_x(0, t) = u_x(L, t) = 0$ for all $t > 0$. This implies that no heat can flow across either end of the rod. The Method of Separation of Variables proceeds exactly as before, except that the boundary conditions on the Sturm-Liouville problem $X'' + \lambda X = 0$ must be changed. With $u(x, t) = X(x)T(t)$, the condition $u_x(0, t) \equiv X'(0)T(t) = 0$, for all $t > 0$ implies that $X'(0) = 0$, and similarly the condition $u_x(L, t) = 0$ implies that $X'(L) = 0$.

To solve the Sturm-Liouville problem $X'' + \lambda X = 0$ with $X'(0) = X'(L) = 0$, we again need to treat the three cases $\lambda < 0$, $\lambda = 0$, $\lambda > 0$.

1. If $\lambda < 0$, let $\lambda = -K^2$. Then the general solution can be written as

$$X(x) = A \cosh(Kx) + B \sinh(Kx),$$

$$X'(x) = AK \sinh(Kx) + BK \cosh(Kx).$$

Using the boundary conditions,

$$X'(0) = AK \sinh(0) + BK \cosh(0) = 0 + BK = 0 \Rightarrow B = 0,$$

$$\text{and } X'(L) = AK \sinh(KL) = 0 \Rightarrow A = 0,$$

since neither K nor L can be zero. This means that $X(x) \equiv 0$ is the *only* solution, and therefore there are no negative eigenvalues.

2. If $\lambda = 0$, then the general solution is $X(x) = C + Dx$, and $X'(x) = D$. Both conditions $X'(0) = X'(L) = 0$ are satisfied if, and only if, the constant $D = 0$. This means that $X(x) \equiv C$, where C is a non-zero constant, is a non-zero solution corresponding to the eigenvalue $\lambda_0 = 0$.

3. If $\lambda > 0$, let $\lambda = K^2$. Then the general solution is

$$X(x) = E \cos(Kx) + F \sin(Kx)$$

$$X'(x) = -KE \sin(Kx) + KF \cos(Kx)$$

and

$$X'(0) = KF = 0 \Rightarrow F = 0.$$

The condition $X'(L) = -KE \sin(KL) = 0$ is satisfied for the infinite sequence of values $KL = \pi, 2\pi, \dots, n\pi, \dots$; therefore, the eigenvalues and corresponding eigenfunctions in this case are

$$\lambda_n = K_n^2 = \frac{n^2\pi^2}{L^2}, \quad X_n(x) = E_n \cos\left(\frac{n\pi x}{L}\right), \quad n = 1, 2, \dots.$$

For each eigenvalue λ_n , $n = 0, 1, \dots$ the equation $T'_n = -\alpha^2\lambda_n T_n$ must be solved (this is the same equation that we found for T_n in the previous lecture). For $\lambda_0 = 0$, the solution $T_0(t)$ is an arbitrary constant; and for $n > 0$, the separable first-order d.e. in T_n has the solution

$$T_n(t) = C_n e^{-\alpha^2\lambda_n t} = C_n e^{-\frac{\alpha^2 n^2 \pi^2 t}{L^2}}.$$

The general solution for u can now be written as

$$u(x, t) = \sum_{n=0}^{\infty} X_n(x) T_n(t) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) e^{-\frac{\alpha^2 n^2 \pi^2 t}{L^2}}. \quad (1)$$

If the initial function is $u(x, 0) = f(x)$, then

$$u(x, 0) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) \equiv f(x).$$

This is exactly in the form of a Fourier Cosine Series for $f(x)$; therefore, the coefficients are

$$A_0 = \frac{a_0}{2} = \frac{1}{2} \cdot \frac{2}{L} \int_0^L f(x) dx, \quad A_n = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx, \quad \text{for } n = 1, 2, \dots. \quad (2)$$

Example 1 Assume a rod of length 10m, with $\alpha^2 = \frac{K}{s\rho} = 1$, is heated to an initial temperature $f(x) = (50 - 2(x - 5)^2)^0 F$. The rod is totally insulated on its ends, as well as around the sides. Determine the temperature $u(x, t)$ in the rod for all $t > 0$.

Using equation (1), we can write the temperature as

$$u(x, t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{10}\right) e^{-\frac{n^2\pi^2 t}{100}},$$

where

$$a_n = \frac{2}{10} \int_0^{10} (50 - 2(x - 5)^2) \cos\left(\frac{n\pi x}{10}\right) dx, \quad n = 0, 1, \dots$$

A MAPLE program was written to compute the coefficients for n from 1 to 30. The resulting graphs of $u(x, t)$ at times $t = 0, 0.5, 2, 5,$ and 20 seconds are shown in Figure (1). Note that, as you would expect, $u(x, t)$ tends to a

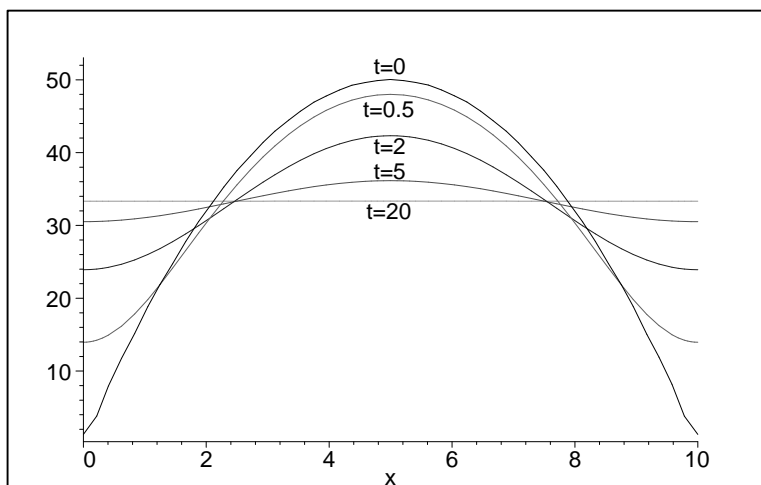


Figure 1: Temperature along the rod at times 0, 0.5, 2, 5, and 20 seconds

constant temperature along the entire rod, where the constant is the average value of the initial temperature function $f(x)$ on the interval $[0, L]$. No heat can escape from the rod.

Non-homogeneous Heat Equation

The term non-homogeneous, in this case, means that the boundary conditions on the heat equation are non-homogeneous (i.e. not both equal to zero).

Here we will solve the one-dimensional heat equation assuming that the temperature at each end of the rod can have arbitrary constant values. To solve this problem, it will be assumed that the temperature $u(x, t)$ is the sum of a function $w(x, t)$ satisfying homogeneous boundary conditions, plus a function $\sigma(x)$ which represents the steady-state temperature in the rod to which $u(x, t)$ tends as $t \rightarrow \infty$; that is, $u(x, t) \equiv w(x, t) + \sigma(x)$.

The function $\sigma(x)$ must satisfy the heat equation $\frac{\partial \sigma}{\partial t} = \alpha^2 \frac{\partial^2 \sigma}{\partial x^2}$, and since σ is not dependant on the time t , $\frac{\partial \sigma}{\partial t} \equiv 0$, implying that $\frac{\partial^2 \sigma}{\partial x^2} = \sigma''(x) \equiv 0$. Therefore, σ is a linear function of x which we can write as

$$\sigma(x) = c_1 + c_2 x.$$

If we assume that the two end temperatures have the constant values $u(0, t) = T_1$ and $u(L, t) = T_2$ for all $t > 0$, then $\sigma(0) = T_1$ and $\sigma(L) = T_2$ completely determines the straight-line function

$$\sigma(x) = T_1 + \frac{T_2 - T_1}{L} x.$$

Now, assuming $u(x, t) = w(x, t) + \sigma(x)$, we can write

$$w(x, t) = u(x, t) - \sigma(x) \tag{3}$$

$$w_t(x, t) = u_t(x, t) - 0 = \alpha^2 u_{xx}(x, t) = \alpha^2 (w_{xx}(x, t) + \sigma''(x)),$$

and since $\sigma''(x) = 0$, we see that the function w also satisfies the heat equation; that is,

$$w_t(x, t) = \alpha^2 w_{xx}(x, t).$$

In addition, $w(0, t) = u(0, t) - \sigma(0) = T_1 - T_1 = 0$ and $w(L, t) = u(L, t) - \sigma(L) = T_2 - T_2 = 0$. This means that $w(x, t)$ is a solution of the heat equation with *homogeneous* boundary conditions $w(0, t) = w(L, t) = 0$ for all $t > 0$. This is the first version of the heat equation that we solved, and the solution was given by

$$w(x, t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) e^{-\frac{\alpha^2 n^2 \pi^2 t}{L^2}}.$$

To find the coefficients b_n , we need to find the initial condition on the function $w(x, t)$. But from equation (3), $w(x, 0) = u(x, 0) - \sigma(x) = f(x) - (T_1 + \frac{T_2 - T_1}{L} x)$; therefore,

$$b_n = \frac{2}{L} \int_0^L \left(f(x) - \left[T_1 + \frac{T_2 - T_1}{L} x \right] \right) \sin\left(\frac{n\pi x}{L}\right) dx, \quad n = 1, 2, \dots .$$

The complete solution of the non-homogeneous problem can now be written in the form

$$u(x, t) = T_1 + \frac{T_2 - T_1}{L}x + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) e^{-\frac{\alpha^2 n^2 \pi^2 t}{L^2}},$$

$$b_n = \frac{2}{L} \int_0^L \left(f(x) - \left[T_1 + \frac{T_2 - T_1}{L}x \right] \right) \sin\left(\frac{n\pi x}{L}\right) dx.$$

Example 2 Assume that a rod of length 4m is heated so that its temperature at time $t = 0$ is given by

$$f(x) = \begin{cases} 0^0 & \text{for } 0 \leq x < 1 \\ 100^0 & \text{for } 1 \leq x < 3 \\ 0^0 & \text{for } 3 \leq x \leq 4 \end{cases}$$

The rod is insulated around the sides and kept constant at temperature $T_1 = 80^0$ at the left end and $T_2 = 20^0$ at the right end. Find the temperature in the rod for all $t > 0$. Assume that $\alpha^2 = \frac{K}{sp} = 0.1$.

The steady state temperature to which the temperature in the rod converges over time is $\sigma(x) = T_1 + \frac{T_2 - T_1}{4}x = 80 - 15x$; therefore,

$$u(x, t) = 80 - 15x + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{4}\right) e^{-\frac{0.1n^2\pi^2 t}{16}}$$

where

$$b_n = \frac{2}{4} \int_0^4 (f(x) - (80 - 15x)) \sin\left(\frac{n\pi x}{4}\right) dx, n = 1, 2, \dots$$

Figure (2) shows the temperature $u(x, t)$ in the rod at times $t = 0, 0.5, 2, 5,$ and 30 . Even with 30 terms in the Fourier Series for $f(x)$, the graph of u at time $t = 0$ is wavy because of the discontinuities at $x = 1$ and $x = 3$; however, as t increases the graphs become much smoother because of the negative exponentials in each term of the series. As $t \rightarrow \infty$, $u(x, t)$ approaches the straight line steady state solution as expected.

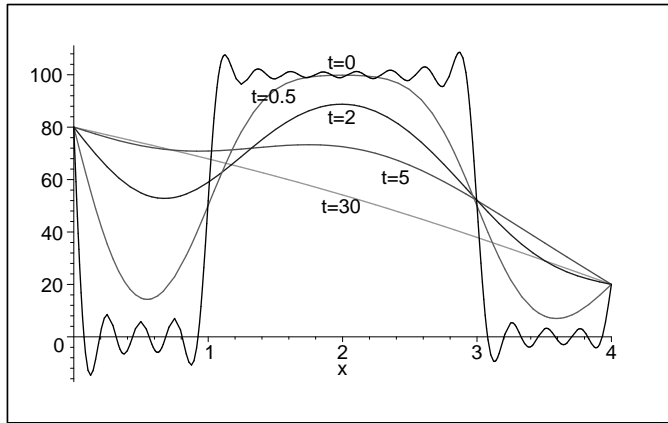


Figure 2: Temperature along the rod at times 0, 0.5, 2, 5, and 30 seconds

Practice Problems:

1. Derive the general solution of the heat equation $u_t = \alpha^2 u_{xx}$ if the end of the rod at $x = 0$ is held constant at 0^0 and the end at $x = L$ is insulated; that is, $u(0, t) = u_x(L, t) = 0$ for all $t > 0$.
2. Solve the heat equation for a rod of length $10m$, with $\alpha^2 = 1$. Assume $u(0, t) = u_x(10, t) = 0$ for $t > 0$, and the initial temperature is given by $f(x) = 2x(10 - x)$, for $0 \leq x \leq 10$. Use MAPLE to compute the coefficients and plot graphs of the temperature along the rod at times $t = 0, 1, 5, 10$, and 20 .
3. In Problem 2, describe in your own words what is happening to the temperature at the right-hand end of the rod. Why does this happen? Use your function $u(x, t)$ to plot a graph of the temperature at $x = 10$ for $0 \leq t \leq 30$.
4. * Solve the heat equation $u_t = 0.5u_{xx}$ with initial temperature $u(x, 0) \equiv 0$ and non-homogeneous boundary conditions $u(0, t) = -20^0$, $u(10, t) = 30^0$ for all $t > 0$. Use MAPLE to plot a graph of the temperature function at times $t = 0, 1, 5, 10$, and 20 .