

M344 - ADVANCED ENGINEERING MATHEMATICS

Lecture 13: Solution of the Heat Equation by Separation of Variables

In the previous lecture we derived the one-dimensional heat equation for the temperature $u(x, t)$ at a point x , at time t , in an insulated rod. We will now assume that the rod is L meters long, totally insulated except for the two ends at $x = 0$ and $x = L$. The density ρ of the rod, its thermal conductivity K , and specific heat s are all assumed to be constant along its length. Under these conditions the temperature $u(x, t)$ in the rod will satisfy the heat equation

$$u_t(x, t) = \alpha^2 u_{xx}(x, t),$$

where α^2 is the positive constant $\frac{K}{s\rho}$.

If this equation is written in the form $\alpha^2 u_{xx}(x, t) - u_t(x, t) = 0$ it can be seen to be a linear second-order p.d.e. with constant coefficients $A = \alpha^2 > 0, B = C = 0$. This means that $B^2 - 4AC = 0$; and, therefore, this is a **parabolic** p.d.e.

In order to obtain a unique solution to such an equation, two types of conditions must be specified.

1. Boundary Conditions:

The temperature $u(x, t)$ must be specified at both ends of the rod, for all values of $t > 0$. There are different ways to do this. One way is to specify the temperature at each end and assume it remains constant for all $t > 0$. Another condition results if it is assumed that one or both of the ends are insulated. If, for example, the end at $x = L$ is insulated, the condition $u_x(L, t) = 0$ for all $t > 0$ is used to simulate the fact that no heat is flowing across that end. One can also model the case where heat is being dissipated at an end by either convection or radiation. We will start by assuming the simplest condition; that is, that the temperature at each end of the rod is held at 0^0 for all $t > 0$.

2. Initial Conditions:

The initial temperature, that is $u(x, 0)$, must be specified as a function $f(x)$ on the interval $0 \leq x \leq L$, where L is the length of the rod. The function $f(x)$ needs to be at least piecewise continuous on $0 \leq x \leq L$.

The problem we are going to solve can be summarized as follows. Find a function $u(x, t)$ such that:

$$\begin{cases} u_t(x, t) = \alpha^2 u_{xx}(x, t), \\ u(0, t) = u(L, t) = 0 \text{ for } t > 0, \\ u(x, 0) = f(x) \text{ for } 0 \leq x \leq L. \end{cases}$$

Method of Separation of Variables

The method of solution we will use is called the **Method of Separation of Variables**. It is first assumed that there exist solutions of the form $u(x, t) = X(x)T(t)$; that is, solutions which are *products* of a function of x times a function of t . There will turn out to be an infinity of such product solutions $u_n(x, t) = X_n(x)T_n(t)$, each of which satisfies the two boundary conditions. Using the fact that the p.d.e. is linear, any linear combination of these, $\sum c_n X_n(x)T_n(t)$ will also be a solution. The constants c_n will then be chosen to make the infinite sum satisfy the initial condition.

If $u(x, t) = X(x)T(t)$, then it is easy to find its partial derivatives. We need u_t and u_{xx} to substitute into the p.d.e.

$$u_t = \frac{\partial}{\partial t} (X(x)T(t)) = X(x)T'(t),$$

where $T'(t)$ denotes the ordinary derivative of $T(t)$ with respect to its single independent variable t . Similarly,

$$u_x = \frac{\partial}{\partial x} (X(x)T(t)) = X'(x)T(t),$$

and, therefore,

$$u_{xx} = \frac{\partial}{\partial x} (X'(x)T(t)) = X''(x)T(t).$$

Substituting these derivatives into the p.d.e. $u_t = \alpha^2 u_{xx}$,

$$X(x)T'(t) = \alpha^2 X''(x)T(t)$$

must be true for $0 \leq x \leq L$ and $t > 0$. The next step is to get everything involving x on one side and everything involving t on the other. This can be done by dividing both sides of the equation by $\alpha^2 X(x)T(t)$:

$$\frac{X(x)T'(t)}{\alpha^2 X(x)T(t)} = \frac{\alpha^2 X''(x)T(t)}{\alpha^2 X(x)T(t)} \implies \frac{T'(t)}{\alpha^2 T(t)} \equiv \frac{X''(x)}{X(x)}.$$

The only way a function of x can be identically equal to a function of t is if both of the functions are equal to the same constant; therefore, we can write

$$\frac{T'(t)}{\alpha^2 T(t)} = \frac{X''(x)}{X(x)} = -\lambda. \quad (1)$$

The reason for using a negative constant is that it will give us a Sturm-Liouville equation to solve for $X(x)$ that looks like one we have already seen. The reason for calling the constant λ is that λ turns out to be an eigenvalue and eigenvalues are conventionally called λ .

Equation (1) can be written in the form of two ordinary differential equations:

$$\begin{cases} T'(t) = -\lambda\alpha^2 T(t), \\ X''(x) + \lambda X(x) = 0. \end{cases}$$

The equation for $X(x)$ turns out to be the Sturm-Liouville equation we discussed in Lecture #11. To obtain boundary conditions on X , we use the given boundary conditions on $u(x, t)$; that is, since $u(x, t) \equiv X(x)T(t)$,

$$\begin{cases} 0 = u(0, t) \equiv X(0)T(t) \\ 0 = u(L, t) \equiv X(L)T(t) \end{cases}$$

has to be true for all $t > 0$. The only way this can be true, without making $T(t)$ identically zero, is to require $X(0) = X(L) = 0$. This leads to a Sturm-Liouville problem for X of the following form:

$$X'' + \lambda X = 0, \quad X(0) = 0, \quad X(L) = 0.$$

This is one of the problems we solved in Lecture #11, and it was shown there that there exists an infinite set of solutions of the form

$$X_n(x) = C_n \sin\left(\frac{n\pi x}{L}\right) \quad (2)$$

corresponding to the eigenvalues $\lambda_n = \frac{n^2\pi^2}{L^2}$, $n = 1, 2, \dots$.

To find the function $T_n(t)$ corresponding to $X_n(x)$, it is necessary to solve the first-order o.d.e.

$$T'_n(t) = -\lambda_n\alpha^2 T_n(t) = -\frac{n^2\pi^2\alpha^2}{L^2} T_n(t).$$

This is a separable first-order equation with general solution

$$T_n(t) = c_n e^{-\frac{n^2\pi^2\alpha^2}{L^2}t}. \quad (3)$$

We now have an *infinite family of solutions* of our heat equation; namely

$$u_n(x, t) = X_n(x)T_n(t) = b_n \sin\left(\frac{n\pi x}{L}\right) e^{-\frac{n^2\pi^2\alpha^2}{L^2}t},$$

where the arbitrary constant b_n is the product of the two arbitrary constants C_n and c_n in equations (2) and (3). Note that each u_n satisfies the two boundary conditions $u_n(0, t) = u_n(L, t) = 0$ for all $t > 0$. So far we have not used the initial condition.

Because the heat equation is linear, linear combinations of solutions are also solutions. If we let

$$u(x, t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) e^{-\frac{n^2\pi^2\alpha^2}{L^2}t},$$

then setting $t = 0$ and using the initial condition $u(x, 0) = f(x)$ results in the requirement that

$$u(x, 0) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \equiv f(x), \quad 0 \leq x \leq L.$$

Recognizing that this is just a Fourier Sine Series for the function $f(x)$ on $0 \leq x \leq L$, the constants b_n must be the coefficients of that Sine Series. Therefore, the complete solution of our heat equation is given by

$$u(x, t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) e^{-\frac{n^2\pi^2\alpha^2}{L^2}t}, \quad b_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx.$$

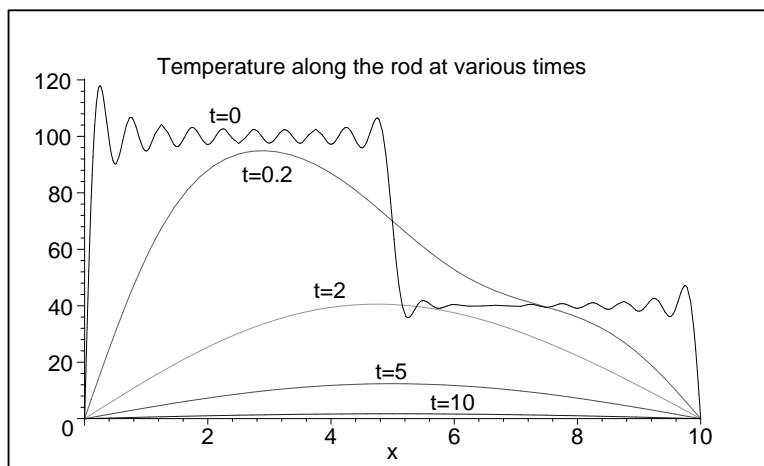
Example 1 *A rod of length 10 meters is initially heated to a temperature of $100^{\circ}F$ on the left half $0 \leq x \leq 5$ and to a temperature of $40^{\circ}F$ on the right half $5 \leq x \leq 10$. If the rod is completely insulated except at the two ends, and the temperature at both ends is held constant at $0^{\circ}F$ for all $t > 0$, find the temperature $u(x, t)$ in the rod for $0 \leq x \leq 10, t > 0$. Assume the constant $\alpha^2 = \frac{K}{s\rho} = 4$.*

We know that the general solution is given by $u(x, t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{10}\right) e^{-\frac{4n^2\pi^2}{100}t}$. Writing the piecewise continuous function $f(x)$ in the form

$$f(x) = \begin{cases} 100 & \text{for } 0 \leq x \leq 5 \\ 40 & \text{for } 5 < x \leq 10 \end{cases}$$

the coefficients are given by

$$\begin{aligned}
 b_n &= \frac{2}{10} \int_0^{10} f(x) \sin\left(\frac{n\pi x}{10}\right) dx = \frac{1}{5} \int_0^5 100 \sin\left(\frac{n\pi x}{10}\right) dx + \frac{1}{5} \int_5^{10} 40 \sin\left(\frac{n\pi x}{10}\right) dx \\
 &= \frac{200}{n\pi} \left(1 - \cos\left(\frac{n\pi}{2}\right)\right) + \frac{80}{n\pi} \left(\cos\left(\frac{n\pi}{2}\right) - \cos(n\pi)\right).
 \end{aligned}$$



The figure above shows a MAPLE plot of the resulting function $u(x, t)$ at five different times $t = 0, 0.2, 2.0, 5.0,$ and 10 . Even though forty terms were used in the Fourier Series it can be seen that the approximation of the initial temperature function at time $t = 0$ is very rough. For $t > 0$ the exponential functions tend to smooth the solution. Note that, as expected, the temperature along the entire rod tends to 0^0 as $t \rightarrow \infty$.

Practice Problems:

- For each p.d.e. below, first determine its type. Then assume the solution can be written as a product $u(x, t) = X(x)T(t)$, and find the o.d.e.s satisfied by $X(x)$ and $T(t)$:
 - $u_t = a^2 u_{xx} + bu_x + cu$ Ans: $T' = \lambda T, a^2 X'' + bX' + (c - \lambda)X = 0$
 - $u_{tt} = bu_{xx}$ Ans: $T'' + \lambda bT = 0, X'' + \lambda X = 0$
 - * $u_{tt} + bu_t + u = u_{xx}$
- * Solve the heat equation in a rod of length 2 meters, if the ends of the rod at $x = 0$ and $x = 2$ are held at 0^0F , and the initial temperature is given by $f(x) = x, 0 \leq x \leq 2$. Assume $\alpha^2 = 1$.

3. * How would the general solution of the heat equation change if we assume the left end of the rod is insulated? This means that the boundary conditions are $u_x(0, t) = 0$, $u(L, t) = 0$. Hint: the homogeneous boundary conditions for the Sturm-Liouville problem $X'' + \lambda X = 0$ change.