M344 - ADVANCED ENGINEERING MATHEMATICS

Lecture 12: Introduction to Partial Differential Equations, Derivation of the Heat Equation

Before learning to solve the partial differential equations encountered in physics and engineering, it is necessary to review the concept of a partial derivative.

Review of Partial Derivatives

In Calculus you learned that if $y(t)$ is a function of one variable t, then the derivative $y'(t)$, can be thought of as the instantaneous rate of change of y at time t. This derivative, when it exists, can be calculated as follows:

$$y'(t) = \frac{d}{dt}(y(t)) = \lim_{\Delta t \to 0} \left(\frac{y(t + \Delta t) - y(t)}{\Delta t} \right).$$

If a function u is a function of more than one variable, say $u = u(x, y)$, then we need to use partial derivatives to measure the rate of change of u with respect to each of the variables. For example, the instantaneous rate of change of u with respect to x is called the **partial derivative of u with respect to x** and is defined by

$$u_x(x, y) \equiv \frac{\partial}{\partial x}(u(x, y)) = \lim_{\Delta x \to 0} \left(\frac{u(x + \Delta x, y) - u(x, y)}{\Delta x} \right).$$

This is just the derivative of u with respect to x, assuming that y is held fixed, and u_x is also a function of both x and y.

Similarly, the **partial derivative of u with respect to y** is given by

$$u_y(x, y) \equiv \frac{\partial}{\partial y}(u(x, y)) = \lim_{\Delta y \to 0} \left(\frac{u(x, y + \Delta y) - u(x, y)}{\Delta y} \right).$$

It will also be necessary to use second-order partial derivatives, and these are defined in the same way. For example, u_{xx} is the instantaneous rate of change of the function $u_x(x, y)$ with respect to the variable x:

$$u_{xx}(x, y) \equiv \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x} \right) = \lim_{\Delta x \to 0} \left(\frac{u_x(x + \Delta x, y) - u_x(x, y)}{\Delta x} \right).$$

It is also denoted by $\frac{\partial^2 u}{\partial x^2}$.

The other second-order partial derivatives are $u_{yy}(x,y) \equiv \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) \equiv \frac{\partial^2 u}{\partial y^2}$ and the mixed derivatives u_{xy} and u_{yx}. For the functions we will be considering, the two mixed partial derivatives are equal, and can be denoted by either $u_{xy} \equiv \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)$ or $u_{yx} \equiv \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)$. The following theorem, which is usually stated in multi-variable Calculus texts, gives the condition under which the two mixed partial derivatives are equal.

Theorem 1 *(Clairaut's Theorem): Suppose u is defined on a region D that contains the point $(x,y) = (a,b)$ in its interior. If the functions u_{xy} and u_{yx} are both continuous on D, then $u_{xy}(a,b) = u_{yx}(a,b)$.*

Example 1 *For the function $u(x,y) = \sin(4x)\cos(3y)$, find all first and second partial derivatives, and show that u satisfies the partial differential equation $u_{xx} - \frac{16}{9}u_{yy} = 0$.*

The six derivatives are

$$u_x = 4\cos(4x)\cos(3y),\quad u_{xx} = -16\sin(4x)\cos(3y),\quad u_{xy} = -12\cos(4x)\sin(3y),$$

$$u_y = -3\sin(4x)\sin(3y),\quad u_{yy} = -9\sin(4x)\cos(3y),\quad u_{yx} = -12\cos(4x)sin(3y).$$

To show that u satisfies the given equation,

$$u_{xx} = -16\sin(4x)\cos(3y) = \frac{16}{9}\left(-9\sin(4x)\cos(3y)\right) = \frac{16}{9}u_{yy};$$

therefore, $u_{xx} - \frac{16}{9}u_{yy} = 0$.

Classification of Linear Second-order Partial Differential Equations

The partial differential equations we will study are all linear equations of second order; where the unknown function u is a function of two independent variables x and y (or x and t if time t is involved, and there is only one space dimension). This means they can be written in the form

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G \qquad (1)$$

where $A, B, \cdots, G$ can be arbitrary functions of x and y. Any equation that can be put in this form is called **linear**, and if $G(x,y) \equiv 0$ it is called a **homogeneous linear p.d.e.**. We will mainly be concerned with homogeneous equations where the functions $A, B, \cdots, F$ are constants. These equations can be classified into three different types, depending on the coefficients of the second-order derivatives, as follows:

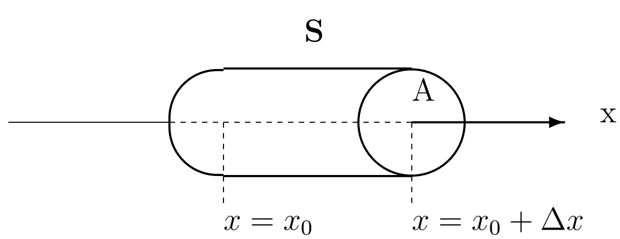
- If $B^2 - 4AC < 0$, equation (1) is called an **elliptic** p.d.e.
- If $B^2 - 4AC = 0$, equation (1) is called a **parabolic** p.d.e.
- If $B^2 - 4AC > 0$, equation (1) is called a **hyperbolic** p.d.e.

We will see that the methods used to solve the equation depend on which type it is. We will study a simple equation of each type. The elliptic equation we will solve is **Laplace's equation** $u_{xx} + u_{yy} = 0$. The **one-dimensional heat equation** $u_t = \alpha^2 u_{xx}$ will be our example of a parabolic p.d.e., and the **wave equation** $u_{tt} = b^2 u_{xx}$ will be the hyperbolic example. All three of these equations, along with several variations of each, appear in many different fields of physics and engineering.

We begin our study of partial differential equations by deriving the one-dimensional heat equation. This will give you an idea of how such an equation can arise. A solution method for this equation will be described in the next lecture.

Derivation of the One-Dimensional Heat Equation $u_t = \alpha^2 u_{xx}$

Let $u(x,t)$ denote the temperature, at time t and position x, along the length of a thin rod with uniform cross-sectional area A, density ρ, and length L. The sides of the rod are perfectly insulated so heat only flows in the x-direction.



Consider a small segment S of the rod. It is known from physics that, if κ is the *thermal conductivity* of the material in the rod, then the heat flow across the face at $x = x_0$ is equal to $-\kappa A u_x(x_0, t)$, and across the face at $x_0 + \Delta x$ the heat flow is $-\kappa A u_x(x_0 + \Delta x, t)$. The negative sign is due to the fact that heat flows from areas of higher temperature to areas of lower temperature, and if

$u_x(x_0, t)$ is positive it means that the temperature is *increasing* in the positive x-direction. We can now write

$$Q \equiv \text{net heat flow into } S \text{ per unit time} = \kappa A[u_x(x_0 + \Delta x, t) - u_x(x_0, t)],$$

and the total amount of heat entering the volume S during the time interval Δt is

$$Q\Delta t = \kappa A[u_x(x_0 + \Delta x, t) - u_x(x_0, t)]\Delta t. \tag{2}$$

Let Δu be the average change in temperature in S over the time interval Δt. The temperature change is known to be proportional to the amount of heat introduced into S and inversely proportional to the mass Δm of S; that is,

$$\Delta u = \frac{Q\ \Delta t}{s\ \Delta m} = \frac{Q\ \Delta t}{s(\ A\Delta x)\ \rho}$$

where the constant of proportionality s is the *specific heat* of the material in the rod. This *average* temperature change occurs at some point inside S, say at $x_0 + \theta\Delta x$, where $0 \leq \theta \leq 1$; therefore,

$$\Delta u = u(x_0 + \theta\Delta x, t + \Delta t) - u(x_0 + \theta\Delta x, t) = \frac{Q\ \Delta t}{s(\ A\Delta x)\ \rho}. \tag{3}$$

Equating the two expressions (2) and (3) for $Q\Delta t$:

$$s\ A\ \Delta x\ \rho[u(x_0+\theta\Delta x, t+\Delta t)-u(x_0+\theta\Delta x, t)] = \kappa A[u_x(x_0+\Delta x, t)-u_x(x_0, t)]\Delta t.$$

Now divide both sides of this equation by $A\Delta x\Delta t$ and let Δx and Δt both approach zero; then

$$\lim_{\Delta x, \Delta t \to 0} s\rho \left[\frac{u(x_0 + \theta\Delta x, t + \Delta t) - u(x_0 + \theta\Delta x, t)}{\Delta t}\right] =$$

$$\lim_{\Delta x, \Delta t \to 0} \kappa \left[\frac{u_x(x_0 + \Delta x, t) - u_x(x_0, t)}{\Delta x}\right].$$

As $\Delta x \to 0$ the point $x_0 + \theta\Delta x \to x_0$ and the above equation becomes

$$s\rho\frac{\partial}{\partial t}u(x_0, t) = \kappa\frac{\partial}{\partial x}\left(u_x(x_0, t)\right);$$

that is, for any values of x and t, $u_t(x, t) = \alpha^2 u_{xx}(x, t)$, where α^2 is the positive constant $\alpha^2 = \frac{K}{s\rho}$.

Practice Problems:

1. Find all first and second partial derivatives of each of the functions below:

   (a) $f(x, y) = e^{2x+3y}$

   Ans: $u_x = 2e^{2x+3y}, u_y = 3e^{2x+3y}, u_{xx} = 4e^{2x+3y}, u_{yx} = 6e^{2x+3y} = u_{xy}, u_{yy} = 9e^{2x+3y}$

   (b) $g(x, y) = xy^4 + 2x^3y + 10x - 5y + 20$

   Ans: $u_x = y^4 + 6x^2y + 10, u_y = 4xy^3 + 2x^3 - 5, u_{xx} = 12xy, u_{xy} = 4y^3 + 6x^2 = u_{yx}, u_{yy} = 12xy^2$

   (c)* $h(x, y) = \sin(4x - 3y)$

2. Classify each of the following partial differential equations as elliptic, parabolic, or hyperbolic. If the equation involves the variables x and t, then replace the variable y in equation (1) by t when applying the test.

   (a) $u_{tt} + 2u_t = 4u_{xx}$ Ans: hyperbolic

   (b) $u_{xx} + 2u_{yy} + u_x + u_y = x + y$ Ans: elliptic

   (c) $u_t = u_{xx} + bu_x + u$ Ans: parabolic

   (d)* $u_{tt} = c^2u_{xx} - bu_t - ku$ (the "telegraph" equation)

3. * Show that the function $u(x, t) = e^{-\alpha^2 t}\sin(x)$ is a solution of the heat equation $u_t = \alpha^2 u_{xx}$.