

M344 - ADVANCED ENGINEERING MATHEMATICS

Lecture 11: Sturm-Liouville Boundary-Value Problems

When solving partial differential equations it will often be necessary to approximate functions by series of orthogonal functions. One way to obtain an orthogonal family of functions is by solving a particular type of boundary-value problem for a second-order linear ordinary differential equation. Up until now you have dealt only with initial-value problems (IVPs) where $x(t_0)$ and $x'(t_0)$ are both given at the same value of t .

Def 1 A non-singular **Sturm-Liouville boundary-value problem** consists of a second-order linear differential equation of the form

$$\frac{d}{dt}(p(t)y'(t)) + (\lambda w(t) - q(t))y(t) = 0, \quad (1)$$

with p, p' , and w continuous functions, q at least piecewise continuous, on the interval $a \leq t \leq b$, and p and w both positive functions on $[a, b]$. The boundary conditions assumed for $y(t)$ are called **homogeneous unmixed boundary conditions**, and are given at two points a and b , in the form:

$$\begin{cases} c_1y(a) + c_2y'(a) = 0 \\ c_3y(b) + c_4y'(b) = 0 \end{cases} \quad (2)$$

There are other types of Sturm-Liouville problems, where the equation has one or more singular points, or the boundary conditions are mixed or periodic, but the above definition describes the type of problem that occurs most often in the partial differential equations we will be solving. For more information on Sturm-Liouville problems you should read the relevant chapter in Nagel & Saff.

The trivial solution $y(t) \equiv 0$ always satisfies (1) and (2), but we are looking for non-zero solutions. It can be shown that non-zero solutions of a Sturm-Liouville problem only exist if the parameter λ belongs to a certain set of real numbers λ_n , called **eigenvalues** of the particular Sturm-Liouville problem. There is a smallest eigenvalue λ_1 , with $\lambda_1 < \lambda_2 < \dots < \lambda_n < \dots$ and $\lim_{n \rightarrow \infty} \lambda_n = \infty$. When $\lambda = \lambda_n$, the corresponding solution $y(t) = \phi_n(t)$ of equation (1) is called an **eigenfunction** corresponding to the eigenvalue λ_n ; and each ϕ_n is unique up to constant multiples.

Theorem 1 The infinite family of eigenfunctions $\mathcal{S} = \{\phi_1, \phi_2, \dots, \phi_n, \dots\}$ of a Sturm-Liouville problem (1) with boundary conditions (2) is an **orthogonal**

family of functions on the interval $[a, b]$, with respect to the weight function $w(t)$ in equation (1); that is

$$\int_a^b \phi_n(t)\phi_m(t)w(t)dt = 0, \text{ if } m \neq n.$$

Example 1 Show that the boundary-value problem

$$y''(t) + \lambda y(t) = 0, \quad y(0) = 0, \quad y'(1) = 0 \quad (3)$$

is a Sturm-Liouville problem, and find all eigenvalues and a corresponding set of orthogonal eigenfunctions.

We can write $y'' + \lambda y = 0$ in the form $(1 \cdot y')' + \lambda \cdot 1 \cdot y = 0$, so this is a Sturm-Liouville equation with $p(t) = w(t) \equiv 1$, and $q(t) \equiv 0$. The given boundary conditions can be written in the form

$$\begin{cases} 1 \cdot y(0) + 0 \cdot y'(0) = 0 \\ 0 \cdot y(1) + 1 \cdot y'(1) = 0 \end{cases} \quad (4)$$

To find all of the non-trivial solutions, we first determine the general solution of the differential equation, and then check to see for which values of λ the given boundary conditions can be satisfied. Since the characteristic equation of the differential equation in (3) is $r^2 + \lambda = 0$, there will be three different cases depending on whether the roots are real and unequal, real and equal, or complex conjugates. We must consider each of these three cases separately.

Case 1: $\lambda < 0$.

Let $\lambda = -K^2$ for some non-zero real number K . Then the roots of the characteristic polynomial $r^2 - K^2 = 0$ are $\pm K$. The general solution in this case can be written as $y(t) = c_1 e^{Kt} + c_2 e^{-Kt}$, but we will find it more convenient to write it in the equivalent form $y(t) = A \cosh(Kt) + B \sinh(Kt)$. Now, to satisfy the two boundary conditions, we need to find A and B such that $y(0) = A \cosh(K \cdot 0) + B \sinh(K \cdot 0) = A = 0$ and $y'(1) = AK \sinh(K \cdot 1) + BK \cosh(K \cdot 1) = 0$. Since A has to be zero, and $K \neq 0$, the condition on $y'(1)$ implies that $B = 0$. This means that the only solution is the trivial solution $y(t) \equiv 0$; therefore, there are no negative eigenvalues.

Case 2: $\lambda = 0$.

In this case, the characteristic polynomial is $r^2 = 0$, with a double root $r = 0$. The general solution is $y(t) = c_1 e^{0 \cdot t} + c_2 t e^{0 \cdot t} = c_1 + c_2 t$, with derivative $y'(t) = c_2$. To satisfy the two boundary conditions, $y(0) = c_1 = 0$ and $y'(1) =$

$c_2 = 0$, so again the only solution is the zero solution; therefore, $\lambda = 0$ is not an eigenvalue.

Case 3: $\lambda > 0$.

Assume $\lambda = K^2$ for some non-zero K . This is the case where the characteristic polynomial $r^2 + K^2 = 0$ has complex conjugate roots $r = \pm Ki$. The general solution in this case is $y(t) = c_1 e^{0 \cdot t} \cos(Kt) + c_2 e^{0 \cdot t} \sin(Kt) = c_1 \cos(Kt) + c_2 \sin(Kt)$ with derivative $y'(t) = -Kc_1 \sin(Kt) + Kc_2 \cos(Kt)$. To satisfy the boundary conditions, we need $y(0) = c_1 = 0$ and $y'(1) = Kc_2 \cos(K) = 0$. Since $K \neq 0$, in order to have $c_2 \neq 0$, it is necessary that $\cos(K) = 0$. This is true for an infinite set of K s, namely

$$K = \frac{\pi}{2}, \frac{3\pi}{2}, \dots, \frac{(2n+1)\pi}{2}, \dots$$

Letting $K_n = \frac{(2n+1)\pi}{2}$, the eigenvalue λ_n is

$$\lambda_n = K_n^2 = \left(\frac{(2n+1)\pi}{2} \right)^2, \quad n = 0, 1, 2, \dots$$

The corresponding eigenfunctions are $y_n(t) = C_n \sin\left(\frac{(2n+1)\pi t}{2}\right)$, $n = 0, 1, \dots$

Note that this gives us a new orthogonal family of functions:

$$\bar{S} = \left\{ \sin\left(\frac{\pi t}{2}\right), \sin\left(\frac{3\pi t}{2}\right), \dots, \sin\left(\frac{(2n+1)\pi t}{2}\right), \dots \right\};$$

which is *not* the family that we used to generate the trigonometric Fourier Series. The family \bar{S} is defined on the interval $[0, 1]$, and since the weight function $w(t)$ in the Sturm-Liouville d.e. $x'' + \lambda x = 0$ is identically equal to one, the functions in \bar{S} satisfy the orthogonality condition

$$\int_0^1 \sin\left(\frac{(2m+1)\pi t}{2}\right) \sin\left(\frac{(2n+1)\pi t}{2}\right) dt = 0$$

if $m \neq n$; and

$$\int_0^1 \left(\sin\left(\frac{(2n+1)\pi t}{2}\right) \right)^2 dt = \frac{1}{2}.$$

Check it!

It can also be shown that any piecewise continuous function $z(t)$, defined on $[0, 1]$ and satisfying the boundary conditions (2), can be expanded in a

convergent series of the form

$$z(t) = \sum_{n=0}^{\infty} a_n \phi_n(t) = \sum_{n=0}^{\infty} a_n \sin\left(\frac{(2n+1)\pi t}{2}\right);$$

and, as shown in Lecture 9, the coefficients in this orthogonal series are given by the formula

$$a_n = \frac{\int_0^1 z(t) \phi_n(t) dt}{\int_0^1 (\phi_n(t))^2 dt} = \frac{\int_0^1 z(t) \sin\left(\frac{(2n+1)\pi t}{2}\right) dt}{\int_0^1 \left(\sin\left(\frac{(2n+1)\pi t}{2}\right)\right)^2 dt} \equiv 2 \int_0^1 z(t) \sin\left(\frac{(2n+1)\pi t}{2}\right) dt.$$

Example 2 For the piecewise continuous function

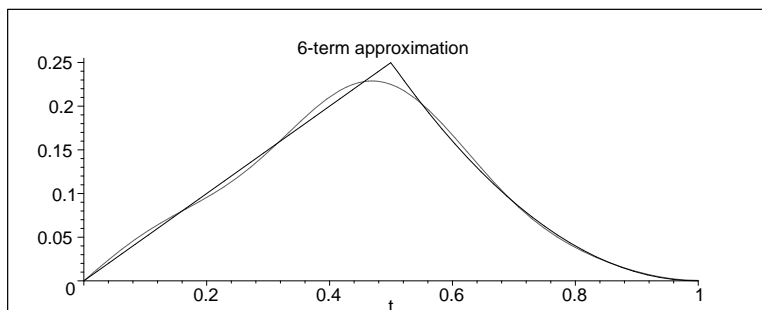
$$z(t) = \begin{cases} 0.5t & \text{if } 0 \leq t \leq 0.5 \\ (t-1)^2 & \text{if } 0.5 < t \leq 1.0 \end{cases}$$

find the first six non-zero terms in the series approximation

$$z(t) \approx \sum_{n=0}^5 a_n \sin\left(\frac{(2n+1)\pi t}{2}\right),$$

and graph the function $z(t)$ together with the finite series approximation on the interval $0 \leq t \leq 1$.

Using the formula $a_n = 2 \int_0^1 (z(t)) \sin\left(\frac{(2n+1)\pi t}{2}\right) dt$, the coefficients a_0, a_1, \dots, a_5 can be found numerically. The graph below shows the function and its approximation on the interval $0 \leq t \leq 1$.



It can be seen that the graph of the series approximation lies very close to the graph of $z(t)$ except at the point where the derivative $z'(t)$ is discontinuous. As you should expect, using more terms in the series will increase the accuracy of the approximation.

Practice Problems:

1. Find all eigenvalues and corresponding eigenfunctions for the Sturm-Liouville problems below:
 - a) $x'' + \lambda x = 0$, $x(0) = 0$, $x(\pi) = 0$.
Ans: $\lambda_n = n^2$, $\phi_n(t) = C_n \sin(nt)$, $n = 1, 2, \dots$.
 - b) $x'' + \lambda x = 0$, $x'(0) = 0$, $x(L) = 0$.
Ans: $\lambda_n = \left(\frac{(2n+1)\pi}{2L}\right)^2$, $\phi_n(t) = C_n \cos\left(\frac{(2n+1)\pi t}{2L}\right)$, $n = 1, 2, \dots$.
 - c) (*) $x'' + \lambda x = 0$, $x'(0) = 0$, $x'(\pi) = 0$.
2. (*) For the Sturm-Liouville problem $x'' + \lambda x = 0$, $x(0) = 0$, $x(1) + x'(1) = 0$,
 - a) Show that there are no negative eigenvalues.
 - b) Show that $\lambda = 0$ is NOT an eigenvalue.
 - c) Show that the positive eigenvalues are of the form $\lambda_n = K_n^2$, where K_n is the n th solution of the equation $\tan(K_n) = K_n$, and that the corresponding eigenfunctions are $\phi_n(t) = C_n \sin(K_n t)$.
 - d) Find numerical values for λ_1 , λ_2 , and λ_3 and graph the corresponding three eigenfunctions ϕ_1 , ϕ_2 and ϕ_3 .
 - e) Find the numerical value of the integral $\int_0^1 \phi_2(t)\phi_3(t)dt$. What should it equal? Why?