Any function \( f(t) \) which is at least piecewise continuous on an interval \([-L, L]\) can be expanded in a Trigonometric Fourier Series

\[
f(t) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi t}{L}\right) + b_n \sin\left(\frac{n\pi t}{L}\right)
\]

\[
a_n = \frac{1}{L} \int_{-L}^{L} f(t) \cos\left(\frac{n\pi t}{L}\right) dt, \quad n = 0, 1, \cdots
\]

\[
b_n = \frac{1}{L} \int_{-L}^{L} f(t) \sin\left(\frac{n\pi t}{L}\right) dt \quad n = 1, 2, \cdots.
\]

We know that piecewise continuous means that \( f \) has at most a finite number of jump discontinuities, say at \( t = t_1, t_2, \cdots, t_n \), and that the limits \( f(t_i^-) = \lim_{t \to t_i^-} f(t) \) and \( f(t_i^+) = \lim_{t \to t_i^+} f(t) \) both exist at each of the points \( t_i \).

**Theorem 1** If \( f(t) \) is a piecewise continuous function on \([-L, L]\), periodic with period \( 2L \), then the Fourier Series (1) for \( f(t) \) converges to \( f(t) \) at every point \( t \) where \( f \) is continuous, and converges to

\[
\frac{f(t_i^+) + f(t_i^-)}{2}
\]

where \( t_i \) is a point of discontinuity.

In the next example we will find a Fourier Series for a function which is specified on a finite interval \([-L, L]\) and assumed to be periodic of period \( 2L \).

**Example 1** Let \( f(t) = t \) on the interval \(-1 \leq t \leq 1\), and assume \( f \) is periodic of period \( 2 \). Draw a graph of \( f \) on the interval \([-3, 3]\). Find the Fourier Series for \( f \), and sketch a graph of the finite sum \( a_0 + \sum_{n=1}^{5} a_n \cos\left(\frac{n\pi t}{L}\right) + b_n \sin\left(\frac{n\pi t}{L}\right) \) which approximates the function \( f \).

Use the formula for \( f \) to sketch a graph of the straight-line function on the interval \([-1, 1]\). To extend it so it is periodic with period \( 2 \), the graph on the intervals \([-3, -1] \) and \([1, 3] \) must look exactly like the graph on \([-1, 1]\).
All of the coefficients \(a_n\) in the Fourier Series are zero, since \(f\) is an odd function on its domain; that is, \(f(-t) = -t = -f(t)\). The coefficients \(b_n\) are found from the formula

\[
b_n = \frac{1}{L} \int_{-L}^{L} f(t) \sin\left(\frac{n\pi t}{L}\right) dt = \frac{2}{L} \int_{0}^{L} f(t) \sin\left(\frac{n\pi t}{L}\right) dt
\]

\[= 2 \int_{0}^{1} t \sin\left(\frac{n\pi t}{L}\right) dt = -\frac{2}{n\pi} \cos(n\pi) .
\]

Figure (1) shows the graph of the periodically extended function \(f(t)\) with the graph of \(\sum_{1}^{5} \left(-\frac{2}{n\pi}(\cos(n\pi))\sin(n\pi t) = \frac{2}{\pi} \left(\frac{\sin(\pi t)}{1} - \frac{\sin(2\pi t)}{2} + \cdots + \frac{\sin(5\pi t)}{5}\right)\right)\) superimposed on it.

![Graph of the function](image)

**Figure 1:**

**Cosine and Sine Series**

When solving partial differential equations, it is often necessary to approximate a given function \(f\) on a **half-interval** \([0, L]\) by a trigonometric series which contains just sine functions, or just cosine functions.

Suppose you are given a function \(f(t)\) defined on \([0, L]\) and want to approximate it on that half-interval by a series of the form

\[
f(t) \sim \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi t}{L}\right).
\]

Since it does not matter what the series converges to on \([-L, 0]\), we can assume that the function \(f\) is extended as an odd function on \([-L, L]\). Then its full Fourier Series will contain only sine terms (the coefficients \(a_n\) are all zero if \(f\) is odd). For \(t \in [0, L]\) the series (2), with \(b_n = \frac{1}{L} \int_{-L}^{L} f(t) \sin\left(\frac{n\pi t}{L}\right) dt = \)
\[ \frac{2}{L} \int_{0}^{L} f(t) \sin \left( \frac{n\pi t}{L} \right) dt \] will converge to \( f(t) \) as desired. Note that we never have to define \( f(t) \) on \([-L, 0]\), but just assume that \( f \) is odd. The series (2) of sine terms is called a **Fourier Sine Series** for \( f(t) \) on \([0, L]\), and the coefficients are

\[ b_n = \frac{2}{L} \int_{0}^{L} f(t) \sin \left( \frac{n\pi t}{L} \right) dt. \]

Similarly, to approximate \( f(t) \) on \([0, L]\) by a series of the form

\[ f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \left( \frac{n\pi t}{L} \right), \]

we can assume that \( f \) is extended to \([-L, L]\) as an even function of period \( 2L \), and use the formula

\[ a_n = \frac{2}{L} \int_{0}^{L} f(t) \cos \left( \frac{n\pi t}{L} \right) dt, \quad n = 0, 1, 2, \ldots . \]

This is called a **Fourier Cosine Series** for \( f(t) \) on \([0, L]\).

In Example 1 we found a full Fourier Series for the function \( f(t) = t \) on \(-1 \leq t \leq 1\). This, in fact, produced a Sine Series for \( f(t) = t \) on \([0, 1]\), since the function \( f(t) = t \) is an odd function on \([-1, 1]\).

**Example 2** Find a Cosine Series for \( f(t) = t \) on \([0, 1]\).

In this case we need to assume that \( f \) is periodic of period \( 2 \) and is extended as an even function on \([-1, 1]\). A graph of \( f(t) \) is shown in Figure (2). To find the coefficients in the Cosine Series

\[ f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \left( \frac{n\pi t}{L} \right), \]

set \( L = 1 \) and compute

\[ a_0 = \frac{2}{L} \int_{0}^{L} f(t) dt, \quad a_n = \frac{2}{L} \int_{0}^{L} f(t) \cos \left( \frac{n\pi t}{L} \right) dt, \quad n = 1, 2, \ldots . \]

Therefore, \( a_0 = 2 \int_{0}^{1} t dt = 1 \) and

\[ a_n = 2 \int_{0}^{1} t \cos(n\pi t) dt = \frac{2}{n^2\pi^2} [\cos(n\pi) - 1] = \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{-4}{n^2\pi^2} & \text{if } n \text{ is odd} \end{cases} \]
The resulting Cosine Series is

\[
f(t) = \frac{1}{2} - \frac{4}{\pi^2} \sum_{n=1,3,...}^{\infty} \frac{\cos(n\pi t)}{n^2} \approx \frac{1}{2} - \frac{4}{\pi^2} \left( \frac{\cos(\pi t)}{1} + \frac{\cos(3\pi t)}{9} + \frac{\cos(5\pi t)}{25} + \cdots \right).
\]

A graph of the periodic extension of \( f(t) \) on \([-3, 3]\), and the sum of the first three terms in the Cosine Series is shown in Figure (2). Note that the Cosine Series converges to \( f(t) \) much more quickly than the Sine Series did. This is because the even extension of the function \( f(t) = t \) is continuous everywhere.

In Figure (1) it can be seen that, at the points of discontinuity of the odd periodic extension of \( f \), the graph of the Fourier Series tends to overshoot the graph of \( f \) on either side of the point of discontinuity. This is an example of Gibb’s phenomenon. You should look this up in a book or on the Web, and read more about it.

Practice Problems:

1. For each function below, sketch a graph of the periodically extended function on the interval \([-3, 3]\). In each case state whether, or not, the graph is continuous on the interval \([-3, 3]\).
   (a) \( f(t) = t^2 \) on \([-1, 1]\).
   (b) \( g(t) = \cos(\frac{\pi}{2} t) \) on \([-1, 1]\).
   (c) \( h(t) = e^t \) on \([-1, 1]\).

2. * Find the first 4 non-zero terms in the Sine Series for the function \( f(t) = t^2 \) on \([0, 1]\). Graph it on the interval \([-3, 3]\).

3. Find the first 4 non-zero terms in the Cosine Series for the function \( q(t) = \sin(t) \) on \([0, \pi]\). Graph it on the interval \([-3\pi, 3\pi]\).
   Answer: \( \sin(t) \approx \frac{2}{\pi} - 0.424413 \cos(2t) - 0.0848826 \cos(4t) - 0.0363783 \cos(6t) \).

4. * Find the first 4 non-zero terms in the Cosine Series for the function \( p(t) = e^t \) on \([0, 1]\). Graph it on the interval \([-3, 3]\).