**M344 - ADVANCED ENGINEERING MATHEMATICS**

**Lecture 10: Properties of Fourier Series, Cosine and Sine Series**

Any function $f(t)$ which is at least piecewise continuous on an interval $[-L, L]$ can be expanded in a Trigonometric Fourier Series

$$\begin{aligned} f(t) &\sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(\frac{n\pi t}{L}) + b_n \sin(\frac{n\pi t}{L}) \\ a_n &= \frac{1}{L}\int_{-L}^{L} f(t)\cos(\frac{n\pi t}{L})dt, \quad n = 0, 1, \cdots \\ b_n &= \frac{1}{L}\int_{-L}^{L} f(t)\sin(\frac{n\pi t}{L})dt \quad n = 1, 2, \cdots . \end{aligned} \tag{1}$$

We know that piecewise continuous means that $f$ has at most a finite number of jump discontinuities, say at $t = t_1, t_2, \cdots, t_n$, and that the limits $f(t_i^-) = \lim_{t\to t_i^-} f(t)$ and $f(t_i^+) = \lim_{t\to t_i^+} f(t)$ both exist at each of the points $t_i$.

**Theorem 1** *If $f(t)$ is a piecewise continuous function on $[-L, L]$, periodic with period $2L$, then the Fourier Series (1) for $f(t)$ converges to $f(t)$ at every point $t$ where $f$ is continuous, and converges to*

$$\frac{f(t_i^+) + f(t_i^-)}{2}$$

*where $t_i$ is a point of discontinuity.*

In the next example we will find a Fourier Series for a function which is specified on a finite interval $[-L, L]$ and assumed to be periodic of period $2L$.

**Example 1** *Let $f(t) = t$ on the interval $-1 \le t \le 1$, and assume $f$ is periodic of period 2. Draw a graph of $f$ on the interval $[-3, 3]$. Find the Fourier Series for $f$, and sketch a graph of the finite sum $\frac{a_0}{2} + \sum_{n=1}^{5} a_n \cos(\frac{n\pi t}{L}) + b_n \sin(\frac{n\pi t}{L})$ which approximates the function $f$.*

*Use the formula for $f$ to sketch a graph of the straight-line function on the interval $[-1, 1]$. To extend it so it is periodic with period 2, the graph on the intervals $[-3, -1]$ and $[1, 3]$ must look exactly like the graph on $[-1, 1]$.*

*All of the coefficients $a_n$ in the Fourier Series are zero, since $f$ is an odd function on its domain; that is, $f(-t) = -t = -f(t)$. The coefficients $b_n$ are found from the formula*

$$b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin(\frac{n\pi t}{L})dt = \frac{2}{L}\int_{0}^{L} f(t)\sin(\frac{n\pi t}{L})dt$$

$$= 2\int_{0}^{1} t\sin(\frac{n\pi t}{1})dt = -\frac{2}{n\pi}(\cos(n\pi)).$$

*Figure (1) shows the graph of the periodically extended function $f(t)$ with the graph of $\sum_1^5(-\frac{2}{n\pi}(\cos(n\pi))sin(n\pi t) = \frac{2}{\pi}(\frac{\sin(\pi t)}{1} - \frac{\sin(2\pi t)}{2} + \cdots + \frac{\sin(5\pi t)}{5})$ superimposed on it.*

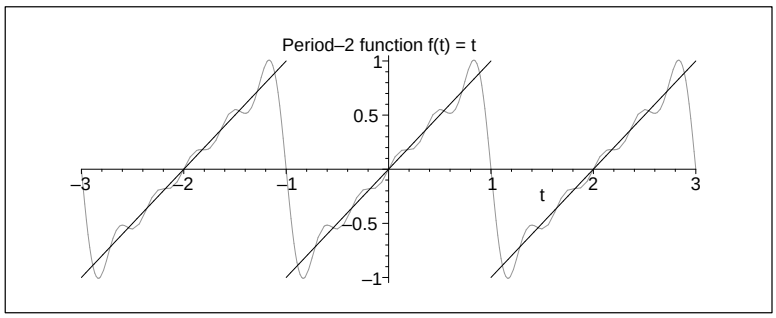


Figure 1:

## Cosine and Sine Series

When solving partial differential equations, it is often necessary to approximate a given function $f$ on a **half-interval** $[0, L]$ by a trigonometric series which contains just sine functions, or just cosine functions.

Suppose you are given a function $f(t)$ defined on $[0, L]$ and want to approximate it on that half-interval by a series of the form

$$f(t) \sim \sum_{n=1}^{\infty} b_n \sin(\frac{n\pi t}{L}). \tag{2}$$

Since it does not matter what the series converges to on $[-L, 0]$, we can assume that the function $f$ is extended as an *odd* function on $[-L, L]$. Then its full Fourier Series will contain only sine terms (the coefficients $a_n$ are all zero if $f$ is odd). For $t \in [0, L]$ the series (2), with $b_n = \frac{1}{L}\int_{-L}^{L} f(t)sin(\frac{n\pi t}{L})dt =$

$\frac{2}{L}\int_0^L f(t)sin(\frac{n\pi t}{L})dt$ will converge to $f(t)$ as desired. Note that we never have to define $f(t)$ on $[-L,0]$, but just assume that $f$ is odd. The series (2) of sine terms is called a **Fourier Sine Series** for $f(t)$ on $[0,L]$, and the coefficients are

$$b_n = \frac{2}{L}\int_0^L f(t)\sin(\frac{n\pi t}{L})dt.$$

Similarly, to approximate $f(t)$ on $[0,L]$ by a series of the form

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(\frac{n\pi t}{L}),$$

we can assume that $f$ is extended to $[-L,L]$ as an *even* function of period $2L$, and use the formula

$$a_n = \frac{2}{L}\int_0^L f(t)\cos(\frac{n\pi t}{L})dt, \quad n = 0,1,2,\cdots.$$

This is called a **Fourier Cosine Series** for $f(t)$ on $[0,L]$.

In Example 1 we found a full Fourier Series for the function $f(t) = t$ on $-1 \le t \le 1$. This, in fact, produced a Sine Series for $f(t) = t$ on $[0,1]$, since the function $f(t) = t$ is an odd function on $[-1,1]$.

**Example 2** *Find a Cosine Series for $f(t) = t$ on $[0,1]$.*

*In this case we need to assume that $f$ is periodic of period 2 and is extended as an even function on $[-1,1]$. A graph of $f(t)$ is shown in Figure (2). To find the coefficients in the Cosine Series*

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(\frac{n\pi t}{L}),$$

*set $L = 1$ and compute*

$$a_0 = \frac{2}{L}\int_0^L f(t)dt, \qquad a_n = \frac{2}{L}\int_0^L f(t)\cos(\frac{n\pi t}{L})dt, \quad n = 1,2,\cdots.$$

*Therefore, $a_0 = 2\int_0^1 tdt = 1$ and*

$$a_n = 2\int_0^1 t\cos(n\pi t)dt = \frac{2}{n^2\pi^2}[\cos(n\pi) - 1] = \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{-4}{n^2\pi^2} & \text{if } n \text{ is odd} \end{cases}$$

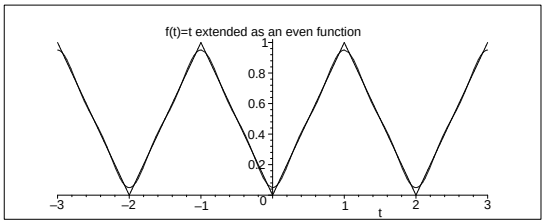


Figure 2:

*The resulting Cosine Series is*

$$f(t) = \frac{1}{2} - \frac{4}{\pi^2} \sum_{n=1,3,\ldots}^{\infty} \frac{\cos(n\pi t)}{n^2} \approx \frac{1}{2} - \frac{4}{\pi^2} \left( \frac{\cos(\pi t)}{1} + \frac{\cos(3\pi t)}{9} + \frac{\cos(5\pi t)}{25} + \cdots \right).$$

*A graph of the periodic extension of $f(t)$ on $[-3, 3]$, and the sum of the first three terms in the Cosine Series is shown in Figure (2). Note that the Cosine Series converges to $f(t)$ much more quickly than the Sine Series did. This is because the even extension of the function $f(t) = t$ is continuous everywhere.*

*In Figure (1) it can be seen that, at the points of discontinuity of the odd periodic extension of $f$, the graph of the Fourier Series tends to overshoot the graph of $f$ on either side of the point of discontinuity. This is an example of* **Gibb's phenomenon**. *You should look this up in a book or on the Web, and read more about it.*

**Practice Problems:**

1. For each function below, sketch a graph of the periodically extended function on the interval $[-3, 3]$. In each case state whether, or not, the graph is continuous on the interval $[-3, 3]$.

   (a) $f(t) = t^2$ on $[-1, 1]$.

   (b) $g(t) = \cos(\frac{\pi}{2}t)$ on $[-1, 1]$.

   (c) $h(t) = e^t$ on $[-1, 1]$.

2. * Find the first 4 non-zero terms in the Sine Series for the function $f(t) = t^2$ on $[0, 1]$. Graph it on the interval $[-3, 3]$.

3. Find the first 4 non-zero terms in the Cosine Series for the function $q(t) = \sin(t)$ on $[0, \pi]$. Graph it on the interval $[-3\pi, 3\pi]$.

   Answer: $\sin(t) \sim \frac{2}{\pi} - 0.424413\cos(2t) - 0.0848826\cos(4t) - 0.0363783\cos(6t)$.

4. * Find the first 4 non-zero terms in the Cosine Series for the function $p(t) = e^t$ on $[0, 1]$. Graph it on the interval $[-3, 3]$.