

M344 - ADVANCED ENGINEERING MATHEMATICS

Lecture 1: Mass-spring Equation, Complex Numbers

In M242 you were shown how the ordinary differential equation

$$mx'' + bx' + kx = 0 \quad (1)$$

models the displacement $x(t)$ at time t of a heavy object suspended on a spring. The positive parameters m , b , and k represent the **mass** of the object, the **damping coefficient**, and the **spring constant**. When m , b , and k are constants, the equation can be solved by assuming a solution of the form $x(t) = e^{rt}$ and substituting x , $x' = re^{rt}$ and $x'' = r^2e^{rt}$ into equation (1) to get

$$mr^2e^{rt} + bre^{rt} + ke^{rt} = e^{rt}(mr^2 + br + k) = 0.$$

The quadratic equation $mr^2 + br + k = 0$ is called the **characteristic equation**, and its roots r_1 and r_2 are the exponents r such that e^{rt} is a solution of the differential equation. Since (1) is a second-order differential equation, the **general solution** consists of a linear combination of two linearly independent solutions. There are three cases to consider:

1. If the roots r_1 and r_2 are real and unequal, $x(t) = c_1e^{r_1t} + c_2e^{r_2t}$.
2. If there is a double real root r , $x(t) = c_1e^{rt} + c_2te^{rt}$.
3. If the roots are complex $\alpha \pm \beta i$, $x(t) = c_1e^{\alpha t} \cos(\beta t) + c_2e^{\alpha t} \sin(\beta t)$.

Using the quadratic formula, the roots of the characteristic equation $mr^2 + br + k = 0$ are

$$r = \frac{-b \pm \sqrt{b^2 - 4mk}}{2m},$$

and therefore,

- Case 1 occurs if $b^2 > 4mk$, and the system is called **over damped**.
- Case 2 occurs if $b^2 = 4mk$, and the system is called **critically damped**.
- Case 3 occurs if $b^2 < 4mk$, and the system is called **under damped**.

In Case 3 it is necessary to find the complex roots $\alpha \pm \beta i$. Note that in this case, $\alpha = \frac{-b}{2m}$, $\beta = \frac{\sqrt{4mk - b^2}}{2m}$, and $\mathbf{i} = \sqrt{-1}$.

Example 1 Find the roots of $r^2 + 2r + 4 = 0$.

Using the quadratic formula, $r = \frac{-2 \pm \sqrt{4 - 4(1)(4)}}{2} = -1 \pm \frac{\sqrt{-12}}{2} = -1 \pm \sqrt{3}\mathbf{i}$.

Complex roots of a quadratic equation can also be found using the TI-89. In MODE, **complex format** must be set to **rectangular**. Once this is done, to find the two roots of $r^2 + 2r + 4 = 0$, go to **catalog** and enter **cSolve**. The expression **cSolve($x^2 + 2x + 4 = 0, x$)** should return the answer

$$x = -1 + \sqrt{3} \mathbf{i} \text{ or } -1 - \sqrt{3} \mathbf{i}.$$

Example 2 Find the solution of the initial-value problem

$$x'' + 2x' + 4x = 0, \quad x(0) = 2, \quad x'(0) = -10.$$

Plot the solution for $0 \leq t \leq 7$ using MAPLE. Describe the behavior of the object on the spring, using the solution $x(t)$ which measures its displacement. Is this system under or over damped?

We found the roots of the characteristic polynomial in Example 1. They are complex, so the formula for Case 3 gives the general solution:

$$x(t) = c_1 e^{-t} \cos(\sqrt{3}t) + c_2 e^{-t} \sin(\sqrt{3}t).$$

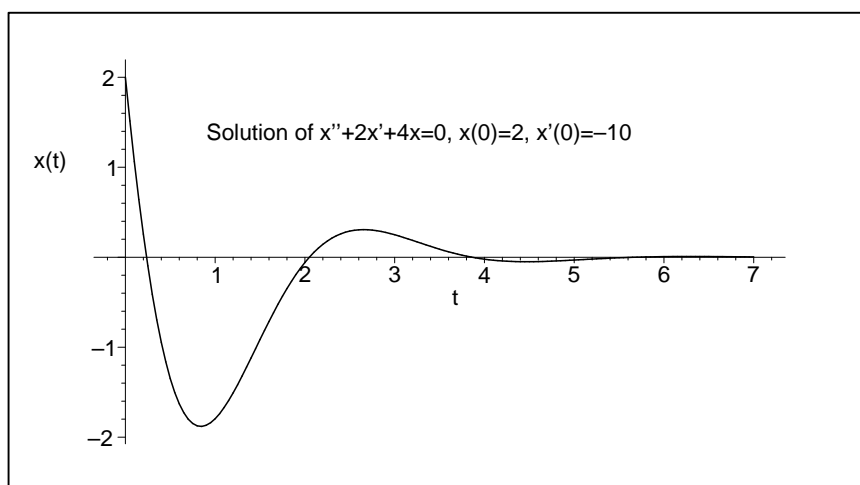
Differentiating this by the product rule gives

$$x'(t) = c_1 \left(-e^{-t} \cos(\sqrt{3}t) - \sqrt{3}e^{-t} \sin(\sqrt{3}t) \right) + c_2 \left(-e^{-t} \sin(\sqrt{3}t) + \sqrt{3}e^{-t} \cos(\sqrt{3}t) \right).$$

Using the initial conditions,

$$x(0) = c_1 = 2, \quad \text{and} \quad x'(0) = -c_1 + \sqrt{3}c_2 = -10 \Rightarrow c_2 = -\frac{8\sqrt{3}}{3}.$$

The general solution is $x(t) = e^{-t} \left(2 \cos(\sqrt{3}t) - \frac{8\sqrt{3}}{3} \sin(\sqrt{3}t) \right)$.



The above graph of the solution is produced in MAPLE by executing the instructions:

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>with(DEtools): with(plots):
> P1:=DEplot({diff(x(t),t$2)+2*diff(x(t),t)+4*x(t)=0}, [x(t)],t=0..7,
             [[x(0)=2,D(x)(0)=-10]],stepsize=0.05,linecolor=BLACK):
> P2:=textplot([3.5,1.4,'Solution of x''+2x'+4x=0, x(0)=2, x'(0)=-10']):
> display(P1,P2);

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This system is under damped, since $b^2 - 4mk = -12 < 0$. The mass oscillates about its equilibrium position $x = 0$, and the oscillation damps out as $t \rightarrow \infty$.

Review of Complex Numbers

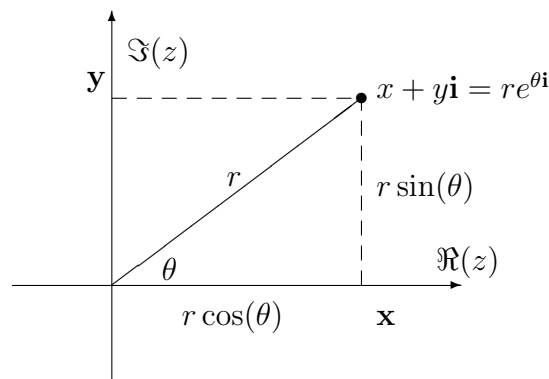
Def 1 A complex number is any number of the form $z = a + b\mathbf{i}$ where a and b are real numbers and $\mathbf{i} = \sqrt{-1}$.

Complex Arithmetic

- Addition and subtraction: $(a + b\mathbf{i}) \pm (c + d\mathbf{i}) = (a \pm c) + (b \pm d)\mathbf{i}$
- Multiplication: $(a + b\mathbf{i}) \cdot (c + d\mathbf{i}) = (ac - bd) + (ad + bc)\mathbf{i}$
- Division: $\frac{a+b\mathbf{i}}{c+d\mathbf{i}} = \frac{a+b\mathbf{i}}{c+d\mathbf{i}} \cdot \frac{c-d\mathbf{i}}{c-d\mathbf{i}} = \frac{ac+bd}{c^2+d^2} + \frac{(bc-ad)\mathbf{i}}{c^2+d^2}$ (defined unless $c+d\mathbf{i} = 0+0\mathbf{i}$).

The complex number $z = x + y\mathbf{i}$ has **real part** $\Re(z) = x$ and **imaginary part** $\Im(z) = y$. The **complex conjugate** of $z = x + y\mathbf{i}$ is defined to be $\bar{z} = x - y\mathbf{i}$.

Complex numbers can be plotted in the **complex plane** as shown below.



If the coordinates in the complex plane are converted from rectangular (x, y) to polar (r, θ) in the usual way, then $r = \sqrt{x^2 + y^2}$ is called the **norm** or **absolute value** of z , and is denoted by $|z|$. The coordinate $\theta = \arctan(y/x)$ is called the **argument** of z , and is denoted by $\arg(z)$. Note that in polar coordinates, $z = x + y\mathbf{i} = r \cos(\theta) + r \sin(\theta)\mathbf{i} = r(\cos(\theta) + \sin(\theta)\mathbf{i}) \equiv re^{i\theta}$,

where we have used **Euler's formula**, $e^{\theta\mathbf{i}} = \cos(\theta) + \sin(\theta)\mathbf{i}$. Euler's formula will be shown to be true when we review Taylor Series.

Multiplication of complex numbers is simpler in polar form than it is in rectangular form. Using the rules for exponents,

$$z \cdot w = re^{\theta\mathbf{i}} \cdot \rho e^{\psi\mathbf{i}} = r\rho e^{(\theta+\psi)\mathbf{i}} = r\rho(\cos(\theta + \psi) + \sin(\theta + \psi)\mathbf{i}).$$

One simply multiplies the absolute values and adds the arguments of the two complex numbers. This is one reason why engineers working with complex numbers often convert them to polar form. The TI-89 can be used to convert complex numbers from rectangular to polar form and from polar to rectangular form. The following example shows how this can be done.

Example 3 Convert the complex number $1 + \mathbf{i}$ into polar form.

Go to *MODE* and set **complex format to polar**. Then execute $1 + \mathbf{i} \Rightarrow z$. This should produce the result $e^{.785398\mathbf{i}}1.4142135$, or $\sqrt{2}e^{\frac{\pi}{4}\mathbf{i}}$.

To check this by hand, the polar form of $1 + \mathbf{i}$ is $re^{\theta\mathbf{i}}$ where $r = \sqrt{x^2 + y^2} = \sqrt{1+1} = \sqrt{2}$ and $\theta = \arctan(y/x) = \arctan(1) = \frac{\pi}{4}$.

Practice Problems:

- Find all roots of the following equations. First use the quadratic formula, and then check the answer using **cSolve** on the TI-89.

(a) $r^2 + r + 1 = 0$	Ans: $-1/2 \pm (\sqrt{3}/2)\mathbf{i}$
(b) $r^2 + 2r + 10 = 0$	Ans: $-1 \pm 3\mathbf{i}$
* (c) $2r^2 + r + 3 = 0$	
- Transform $2 - 5\mathbf{i}$ into polar form. Ans: $e^{-1.19029\mathbf{i}}5.38517$
- Transform $2e^{\frac{\pi}{3}\mathbf{i}}$ into rectangular form. Ans: $1 + \sqrt{3}\mathbf{i}$
- * Let $z = 4 - 3\mathbf{i}$ and $w = 1 + 2\mathbf{i}$. Find the product $z \cdot w$ and convert it to polar form; then convert z and w into polar form and find $z \cdot w$. Do it by hand and show your work.
- Write out the general solution for each of the following:

(a) $x'' + x' + x = 0$	Ans: $e^{-\frac{1}{2}t}(c_1 \cos(\frac{\sqrt{3}}{2}t) + c_2 \sin(\frac{\sqrt{3}}{2}t))$
(b) $x'' + 2x' + 10x = 0$	Ans: $e^{-t}(c_1 \cos(3t) + c_2 \sin(3t))$
* (c) $2x'' + x' + 3x = 0$	
- * Solve the initial-value problem $x'' + bx' + 4x = 0$, $x(0) = 1$, $x'(0) = 0$ for the 3 cases $b = 2, 4$, and 6 . Use MAPLE to plot the 3 solutions on the same set of axes, from $t = 0$ to $t = 5$, and label each one with the correct value of b . Explain what happens to the motion of the mass as the damping increases.

7. * The mass-spring equation (1) is obtained by using Newton's 2nd Law, mass times acceleration equals sum of forces, so that the force due to the spring is represented by $-kx$. This is Hooke's Law for a "linear" spring. A more general equation, where the spring force is assumed to be slightly *nonlinear*, is called Duffing's equation. It can be written in the form

$$x'' + \delta x' + (\beta x + \alpha x^3) = \gamma \cos(\omega t).$$

If it is modelling a mass-spring system, the constant β is positive. If $\alpha > 0$ the equation represents a mass-spring model with a "hard" spring, and if $\alpha < 0$ it represents a model with a "soft" spring. Duffing's equation can no longer be solved by assuming $x(t) = e^{rt}$, because of the x^3 term. We are going to be able to obtain a series solution of the form $x(t) = \sum_{n=0}^{\infty} a_n t^n$; however, at this point, you can solve the equation numerically using the `DEplot` instruction in MAPLE.

Note: The form of the `DEplot` instruction is given at the top of page 3.

In each problem below assume $x(0) = x'(0) = 0$.

- a) Use `DEplot` to obtain a graph of $x(t)$ where

$$x'' + 0.2x' + (x + x^3) = 0.3 \cos(t), 0 \leq t \leq 80 \quad (\text{hard-spring model})$$

- b) Use `DEplot` to obtain a graph of $x(t)$ where

$$x'' + 0.2x' + (x - x^3) = 0.3 \cos(t), 0 \leq t \leq 80 \quad (\text{soft-spring model})$$

- c) Change the sign of the spring force in (b) and use `DEplot` to obtain a graph of $x(t)$ if

$$x'' + 0.2x' - (x - x^3) = 0.3 \cos(t), 0 \leq t \leq 80 \quad (\alpha < 0)$$

- d) Explain in your own words the major difference between the displacement $x(t)$ in the graph in part (c), as compared to the graphs in (a) and (b).