M344 - ANSWERS TO SAMPLE FINAL (Fall 2006)

1. (20pts) Match exactly one item (a) - (j) with each item in the left-hand column.

..*f*.. i. Parabolic p.d.e.	a) $y'' + by' + ke^{-0.1t}y = 0$
..*c*.. ii. Solution of $u_t = \alpha^2 u_{xx}$	b) $3x^2y'' + xy' - 4y = 0$
..*e*.. iii. $\sinh(x)$	c) $\sum_1^\infty B_n \sin(\frac{n\pi x}{L})e^{-(n\pi\alpha/L)^2 t}$
..*i*.. iv. Zeros of $\cos(x)$	d) $\frac{a_0}{2} + \sum_{n=1}^\infty a_n \cos(\frac{n\pi x}{L})$
..*g*.. v. Maclaurin Series for $\sin(x)$	e) $\frac{e^x - e^{-x}}{2}$
..*a*.. vi. Aging spring equation	f) $u_t = \alpha^2 u_{xx}$
..*b*.. vii. Cauchy-Euler d.e.	g) $\sum_0^\infty (-1)^n \frac{x^{2n+1}}{(2n+1)!}$
..*h*.. viii. d.e. with irreg. sing. point at 0	h) $2x^2y'' + 3y' + x^2y = 0$
..*d*.. ix. Fourier Cosine Series	i) $(2n+1)\pi/2, n = 0, 1, 2, ...$
..*j*.. x. Bessel's equation of order n	j) $x^2y'' + xy' + (x^2 - n^2)y = 0$

2. (20pts) Let $f(x) = \begin{cases} 2 & \text{if } -2 \le x < 0 \\ -2 & \text{if } 0 \le x \le 2 \end{cases}$, and assume f is periodic of period 4.

a) Is the function f even, odd, or neither. Justify your answer.

b) Sketch a graph of $f(x)$ on the interval $-6 \le x \le 6$.

c) If $f(x)$ is expanded in a Fourier Series,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{2}\right) + b_n \sin\left(\frac{n\pi x}{2}\right)\right),$$

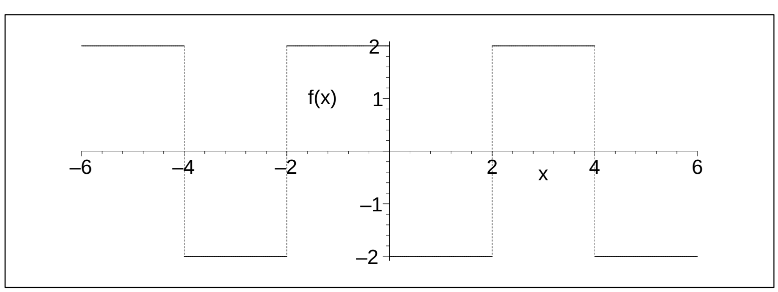
what value will the series converge to at $x = -1$? $x = 2$? $x = 5$?

d) Find the coefficients b_n in the Fourier Series for $f(x)$.

Answers:

a) $f(x)$ is an odd function

b)



c) It will converge to 2 at $x = -1$, to 0 at $x = 2$, and to -2 at $x = 5$.

d) $b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin(\frac{n\pi x}{2})dx = 0$ if n is even, and $-\frac{8}{n\pi}$ if n is odd.

3. (20pts) To find the steady-state temperature inside a spherical tank, it is necessary to convert the Laplacian into spherical coordinates rho, phi and theta. If the boundary conditions are independent of theta, Laplace's equation becomes

$$r^2 u_{rr} + 2r u_r + u_{tt} + \cot(t) u_t = 0.$$

(where rho is denoted by r, and theta by t, to simplify the notation).

a) Let $u(r,t) = R(r)T(t)$, separate the variables, and find ordinary differential equations for $R(r)$ and $T(t)$.

b) Show that the equation for R can be expressed in the form $r^2R''+2rR'-\lambda R = 0$. What type of equation is this; that is, how do you find its solutions?

Answers:

a) The two equations are $r^2R'' + 2rR' - \lambda R = 0$ and $T'' + \cot(t)T' + \lambda T = 0$.

b) $r^2R'' + 2rR' - \lambda R = 0$ is a Cauchy-Euler d.e. It can be solved by letting $R(r) = r^n$. The indicial equation for n is $n^2 + n - \lambda = 0$, and its roots are $n = \frac{-1\pm\sqrt{1+4\lambda}}{2}$.

<u>Extra Credit</u>: Find the value of the constant λ such that the characteristic equation for R has two equal roots. What is the general solution in this case?

Answer:

There will be two equal roots if $\lambda = -\frac{1}{4}$. In this case, $n = -\frac{1}{2}$, and the general solution is $R(r) = c_1 r^{-\frac{1}{2}} + c_2 r^{-\frac{1}{2}} \ln(r)$.

4. (20pts) For the initial-value problem

$$y'' + xy' + 4y = 0, \quad y(0) = 1, \quad y'(0) = 0,$$

a) Is $x = 0$ an ordinary point or a singular point for this differential equation?

b) If $y(t) = \sum_{n=0}^{\infty} a_n x^n$, and its derivatives, are substituted into the equation, it becomes

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + x\sum_{n=1}^{\infty} n a_n x^{n-1} + 4\sum_{n=0}^{\infty} a_n x^n \equiv 0.$$

What are the values of the coefficients a_0 and a_1?

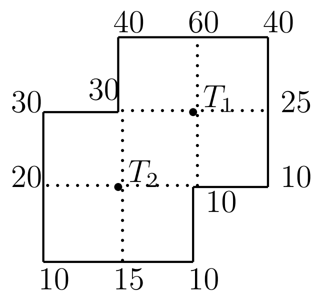
c) Make the necessary changes in the indices of summation, and **find the recurrence formula** for the coefficients a_n.

Answers:

a) $x = 0$ is an ordinary point.

b) The recurrence formula is $a_{m+2} = -\frac{(m+4)a_m}{(m+1)(m+2)}$.

5. (20pts) Using the numerical method we studied in class, find approximate values for the two temperatures T_1 and T_2 shown in the diagram below. It is assumed that the temperature inside satisfies Laplace's equation $u_{xx} + u_{yy} = 0$. The temperatures on the boundary of the region are specified in the diagram.



Answer:

$T_1 = 31.25, T_2 = 18.75$