Introduction

Question 1

. For the function $f(x) = \sin(x)$:

a. Use the forward difference formula with $\Delta x = 0.01$ to estimate $f'(0)$.

b. How small must Δx be in order for the forward difference approximation of $f'(0)$ to be correct to three decimal places.

c. Find the centered difference approximation to $f'(0)$ with $\Delta x = 0.01$.

d. Find the centered difference approximation to $f''(0)$ with $\Delta x = 0.01$.

Solution

Note: $f'(x) = \cos(x)$, so $f'(0) = 1$.

a. $f'(0) \approx \dfrac{f(0+\Delta x) - f(0)}{\Delta x} \approx \dfrac{f(0+0.01) - f(0)}{0.01} = \dfrac{\sin(0.01) - \sin(0)}{0.01} \approx .99998$

b. If we take $\Delta x = 0.05$, we get $f'(0) \approx \dfrac{f(0+0.05) - f(0)}{0.05} = \dfrac{\sin(0.05) - \sin(0)}{0.05} \approx .9995$

c. $f'(0) \approx \dfrac{f(0+\Delta x) - f(0-\Delta x)}{2\cdot\Delta x} = \dfrac{\sin(.01) - \sin(-0.01)}{.02} \approx .999983$

d. $f''(0) \approx \dfrac{f(0+\Delta x) - 2\cdot f(x) + f(0-\Delta x)}{(\Delta x)^2} = \dfrac{\sin(0.01) - 2\cdot\sin(0) + \sin(-.01)}{(0.01)^2} = 0$

Question 2

. a. Solve the heat equation in a wire of length 2 meters, if the temperature at each end is held at 0^o F and the initial temperature is given by $f(x) = \sin\left(\frac{\pi}{2}\cdot x\right), 0 \le x \le 2$. Assume that $\alpha^2=1$. [Hint: the series solution is finite; that is, most terms are 0.]

b. Make a sketch of the temperature in the middle of the wire over the time interval $0 \le t \le 5$.

Solution

a. We know that the formal solution to this problem is

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\cdot\pi\cdot x}{L}\right)\cdot e^{-\frac{n^2\cdot\pi^2\cdot\alpha^2}{L^2}t}, \text{ where } b_n = \frac{2}{L}\int_0^L f(x)\cdot\sin\left(\frac{n\cdot\pi\cdot x}{L}\right)dx$$

In this case, $\alpha^2 = 1$, $L = 2$, and $f(x) = \sin\left(\frac{\pi}{2}\cdot x\right)$. Compute b_n.

$$b_n = \frac{2}{2}\int_0^2 \sin\left(\frac{\pi}{2}\cdot x\right)\cdot\sin\left(\frac{n\cdot\pi\cdot x}{2}\right)dx = -\frac{2\sin(n\pi)}{\pi(-1+n^2)} = 0 \text{ when } n \ne 1.$$

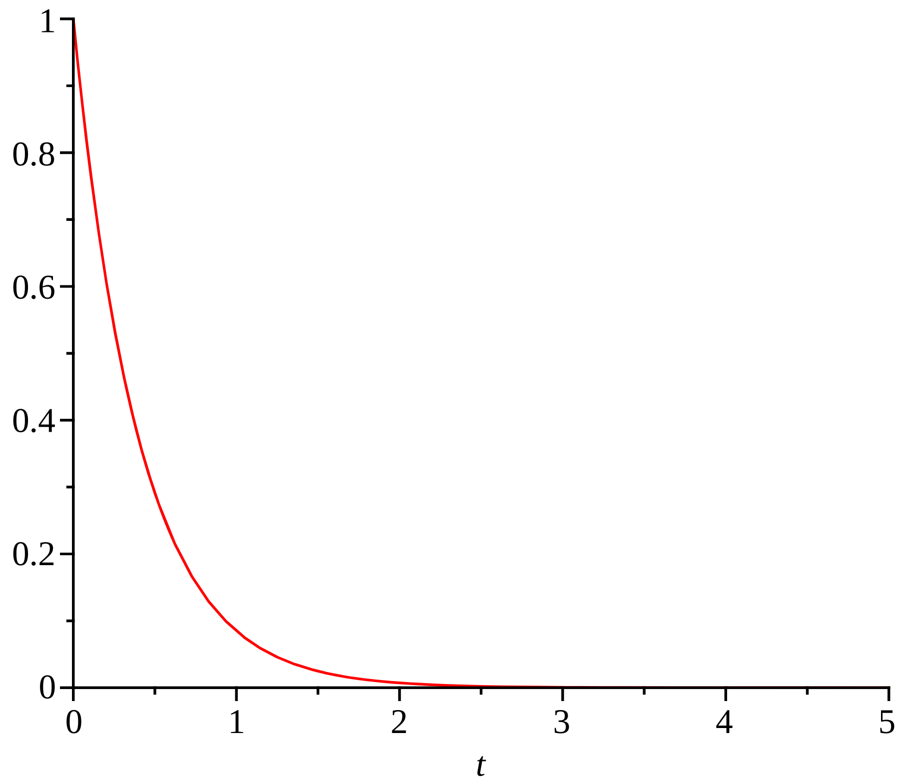
$$b_1 = \frac{2}{2}\int_0^2 \sin\left(\frac{\pi}{2}\cdot x\right)\cdot\sin\left(\frac{\pi\cdot x}{2}\right)dx = 1.$$

Therefore, the solution is

$$u(x,t) = \sin\left(\frac{\pi}{2}\cdot x\right)\cdot e^{-\frac{\pi^2}{4}\cdot t}.$$

b.

$$plot\left(\sin\left(\frac{\pi}{2}\cdot 1\right)\cdot e^{-\frac{\pi^2}{4}\cdot t}, t = 0..5\right)$$



Question 3

Suppose you have a wire 1 meter long with insulated ends whose initial temperature is given by $f(x) = \sin(\pi\cdot x), 0 \le x \le 1$. Assuming α^2=1,what is the temperature in the wire after a long time?

Solution

The formal solution to this problem is

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\cdot\pi\cdot x}{L}\right)\cdot e^{-\frac{n^2\cdot\pi^2\cdot\alpha^2}{L^2}t} \quad \text{where } a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\,\pi\,x}{L}\right)dx$$

If we let $t \to \infty$, $u(x,t) \to \frac{a_0}{2} = \frac{2}{1}\int_0^1 \sin(\pi\cdot x)\cos\left(\frac{0\cdot\pi\cdot x}{1}\right)dx = \frac{2}{1}\int_0^1 \sin(\pi\cdot x)\,dx = \frac{4}{\pi}$.

Question 4

a. Find the D'Alembert solution to the wave equation

$$u_{tt}(x,t) = \frac{1}{4}\cdot u_{xx}(x,t)$$

$$u(x,0) = \begin{cases} 8\cdot x\cdot(1-x) & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$u_t(x, 0) = 0$

b. Graph the solution for $-2 \le x \le 2$ at times $t = 0$, $t = \frac{1}{2}$, and $t = 1$. Use a separate graph for each time.

Solution

a. D'Alembert's solution is $u(x,t) = \frac{1}{2}\cdot\left(f(x+\alpha\cdot t) + f(x-\alpha\cdot t)\right) + \frac{1}{2\cdot\alpha}\cdot\int_{x-\alpha\cdot t}^{x+\alpha\cdot t} g(u)\,du$

where $f(x) = u(x,0) = \begin{cases} 8\cdot x\cdot(1-x) & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$ and $g(x) = u_t(x,0) = 0$. In this case, $\alpha = \frac{1}{2}$. So, $u(x,t) = \frac{1}{2}\cdot\left(f\left(x+\frac{1}{2}\cdot t\right) + f\left(x-\frac{1}{2}\cdot t\right)\right)$.

b.

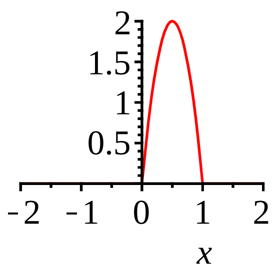
$$f(x) := \begin{cases} 8\cdot x\cdot(1-x) & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases} :$$

$$u(x,t) := \frac{1}{2}\cdot\left(f\left(x+\frac{1}{2}\cdot t\right) + f\left(x-\frac{1}{2}\cdot t\right)\right);$$

$$u := (x,t) \mapsto \frac{f\left(x+\frac{t}{2}\right)}{2} + \frac{f\left(x-\frac{t}{2}\right)}{2} \qquad \textbf{(5.1.1)}$$

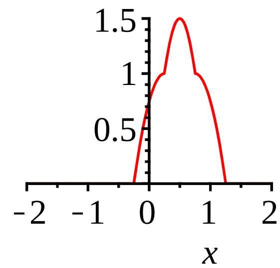
Graph of $u(x,t)$ for $t = 0$.

$plot(u(x,0), x = -2..2)$

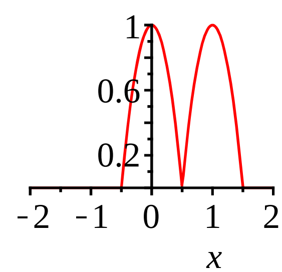


Graph of $u(x,t)$ for $t = \frac{1}{2}$.

$plot\left(u\left(x, \frac{1}{2}\right), x = -2..2\right);$

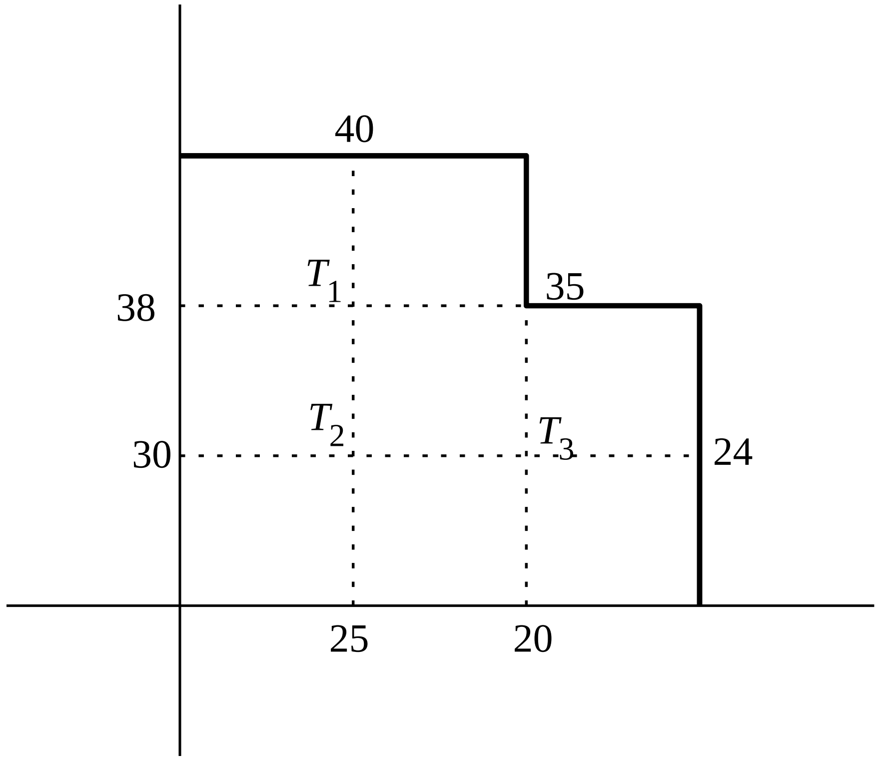


Graph of $u(x,t)$ for $t = 1$.
$plot(u(x,1), x = -2..2);$



Question 5

Use the numerical method for solving Laplace's equation to find approximations to T_1, T_2, and T_3 in the L-shape region shown below. Show all steps.



Solution

$eq1 := 4\,T_1 = 38 + 40 + 35 + T_2:$

$eq2 := 4\,T_2 = 30 + T_1 + T_3 + 25:$

$eq3 := 4\,T_3 = T_2 + 35 + 24 + 20:$

$A := GenerateMatrix(\{eq1, eq2, eq3\}, \{T_1, T_2, T_3\}, augmented = true)$

$$A := \begin{bmatrix} 4 & -1 & 0 & 113 \\ 0 & -1 & 4 & 79 \\ -1 & 4 & -1 & 55 \end{bmatrix} \quad \textbf{(6.1.1)}$$

$evalf(rref(A))$

$$\begin{bmatrix} 1. & 0. & 0. & 35.60714286 \\ 0. & 1. & 0. & 29.42857143 \\ 0. & 0. & 1. & 27.10714286 \end{bmatrix} \quad \textbf{(6.1.2)}$$