M344 Review Questions for Exam 2

Question 1

1. a. Show that the functions $p_0(x) = 1$, $p_1(x) = x$, $p_2(x) = x^2 - \frac{1}{3}$ and $p_3(x) = x^2 - \frac{1}{3}$ are an orthogonal family
   on the interval [-1, 1].
   b. Find the orthogonal expansion of the function $f(x) = \sin(\pi \cdot x)$.

Solution

$p_0 := 1 :\ p_1 := x :\ p_2 := x^2 - \frac{1}{3} :$

$$\int_{-1}^{1} p_0 \cdot p_1 \, dx$$

$$0 \qquad \textbf{(1.1.1)}$$

$$\int_{-1}^{1} p_0 \cdot p_2 \, dx$$

$$0 \qquad \textbf{(1.1.2)}$$

$$\int_{-1}^{1} p_1 \cdot p_2 \, dx$$

$$0 \qquad \textbf{(1.1.3)}$$

The orthogonal expansion is

$$\frac{\int_{-1}^{1} p_0 \cdot \sin(\pi \cdot x) \, dx}{\int_{-1}^{1} p_0^2 \, dx} \cdot p_0 + \frac{\int_{-1}^{1} p_1 \cdot \sin(\pi \cdot x) \, dx}{\int_{-1}^{1} p_1^2 \, dx} \cdot p_1 + \frac{\int_{-1}^{1} p_2 \cdot \sin(\pi \cdot x) \, dx}{\int_{-1}^{1} p_2^2 \, dx} \cdot p_2$$

$$\frac{3\,x}{\pi} \qquad \textbf{(1.1.4)}$$

Question 2

2. Given the function $f(x) = \begin{cases} 0 & -2 \le x < 1 \\ x^2 & -1 \le x < 1 \\ 0 & 1 \le x \le 2 \end{cases}$, f periodic with period 4

   a. Sketch a graph of $f(x)$ on the interval $-6 \le x \le 6$.
   b. Is f an even function, odd function or neither? Justify your answer.

c. What value will the Fourier series for $f(x)$ converge to for $x=-1$? For $x=2$? For $x=\frac{1}{2}$?

d. What is the value of the constant $\frac{a_0}{2}$ in the Fourier series?

Solution

restart;

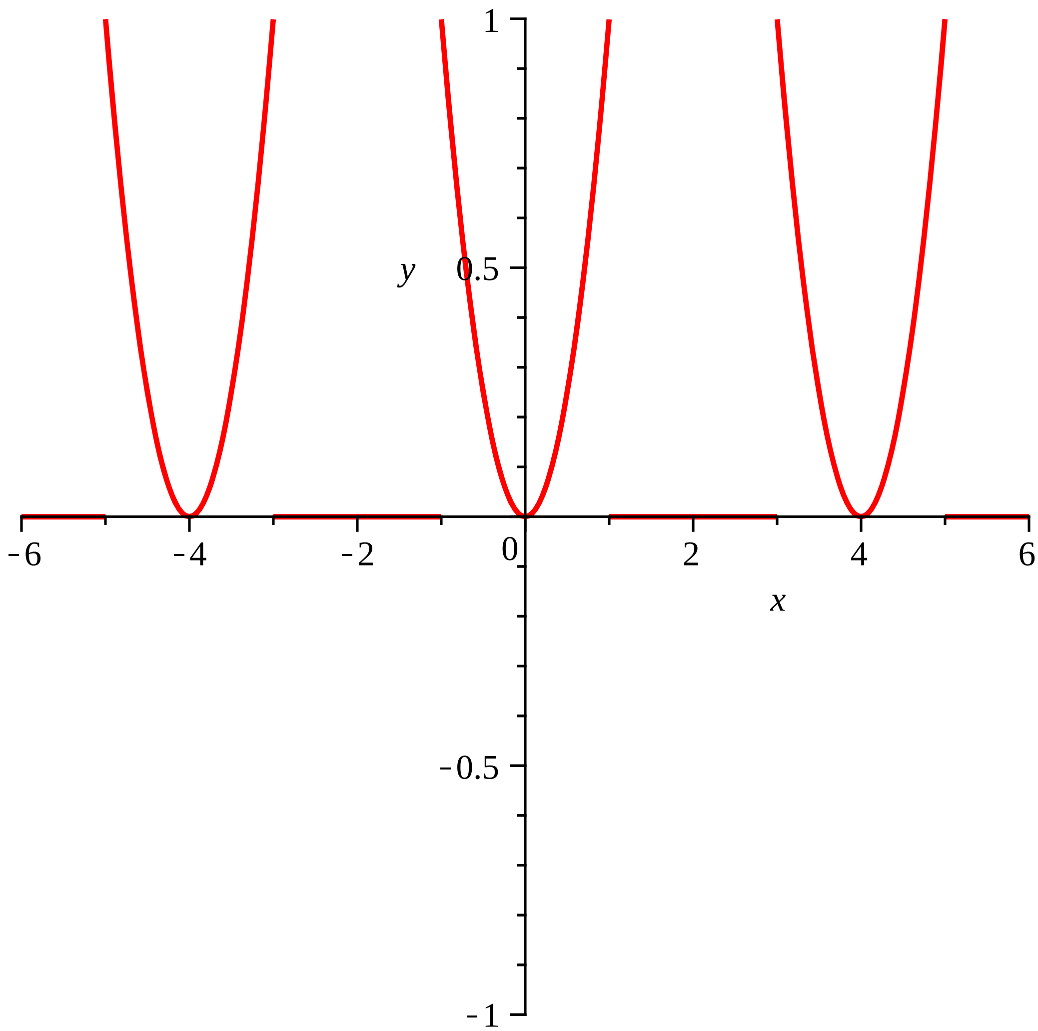
a. The plot is

$f(x) := piecewise(-2 \le x \textbf{ and } x < -1, 0, -1 \le x \textbf{ and } x < 1, x^2, 1 \le x \textbf{ and } x \le 2, 0);$

$$f := x \mapsto \begin{cases} 0 & -2 \le x < -1 \\ x^2 & -1 \le x < 1 \\ 0 & 1 \le x \le 2 \end{cases} \qquad \textbf{(2.1.1)}$$

$g := f\left(x - 4\cdot \text{floor}\left(\frac{x+2}{4}\right)\right):$

$plot(g, x=-6..6, y=-1..1, discont=true, thickness=2);$



b. The function is even because it is symmetric across the y-axis.

c. For $x=-1$, the Fourier series will converge to

$$\frac{f(x^-)+f(x^+)}{2} = \frac{0+1}{2} = \frac{1}{2}.$$

For $x = 2$, the Fourier series converges to $f(2) = 0$.

For $x = \frac{1}{2}$, the series converges to $f\left(\frac{1}{2}\right) = \frac{1}{4}$.

d. $a_0 = \frac{1}{2} \cdot \int_{-2}^{2} f(x)\, dx = a_0 = \frac{1}{3}$. So $\frac{a_0}{2} = \frac{1}{6}$. Note: this is the average value of the function $f(x)$ on the interval $[-2, 2]$.

Question 3

3. Given the partial differential equation $u_t(x,t) = \alpha^2 \cdot u_{xx}(x,t) + 2 \cdot u_x(x,t)$.
   a. Let $u(x,t) = X(x) \cdot T(t)$ and find ordinary differential equations satisfied by $X(x)$ and $T(t)$.
   b. If the boundary conditions on the partial differential equation are $2 \cdot u(0,t) - u_x(0,t) = 0$ and $u(L,t) = 0$ for $t > 0$, find boundary conditions for the ordinary differential equation for $X(x)$.

Solution

a. $u(x,t) = X(x) \cdot T(t)$ $\quad u_x = X'(x) \cdot T(t)$ $\quad u_{xx} = X''(x) \cdot T(t)$ $\quad u_t = X(x) \cdot T'(t)$

Substituting into the pde gives

$$X(x) \cdot T'(t) = \alpha^2 \cdot X''(x) \cdot T(t) + 2X'(x) \cdot T(t) = T(t) \cdot \left(\alpha^2 \cdot X''(x) + 2 \cdot X'(x)\right)$$

Separate

$$\frac{T'(t)}{T(t)} = \frac{\alpha^2 \cdot X''(x) + 2 \cdot X'(x)}{X(x)} = -\lambda$$

So we get the two ordinary differential equations

$$T'(t) = -\lambda \cdot T(t)$$
$$\alpha^2 \cdot X''(x) + 2 \cdot X'(x) + \lambda \cdot X(x) = 0$$

b. $2 \cdot u(0,t) - u_x(0,t) = 2 \cdot X(0) \cdot T(t) - X'(0) \cdot T(t) = (2 \cdot X(0) - X'(0)) \cdot T(t) - 0$. This must be true for all t, so one boundary condition for X is

$$2 \cdot X(0) - X'(0) = 0$$

$u(L,t) = X(L) \cdot T(t) = 0$ gives the other boundary condition $X(L) = 0$.

Question 4

4. Find all eigenvalues λ_n and the corresponding eigenfunctions $X_n(x)$ for the boundary value problem

$$X'' + \lambda \cdot X = 0,\ \ X(0) = 0,\ \ X'(4) = 0$$

Do all three cases, $\lambda < 0$, $\lambda = 0$, $\lambda > 0$.

Solution

$\lambda < 0$. $\lambda = -K^2$. The general solution is $X = c_1 \cdot \cosh(K \cdot x) + c_2 \cdot \sinh(K \cdot x)$.

$X(0) = c_1 \cosh(0) = c_1 = 0$

$X'(x) = c_1 \cdot K \cdot \sinh(K \cdot x) + c_2 \cdot K \cdot \cosh(K \cdot x) = c_2 \cdot K \cdot \cosh(K \cdot x)$ because $c_1 = 0$.

$X'(4) = c_2 \cdot K \cdot \cosh(4 \cdot K) = 0$. This implies $c_2 = 0$ because $K \neq 0$ and $\cosh(4 \cdot K) \neq 0$

So there are no negative eigenvalues.

$\lambda = 0$. The general solution is $X = c_1 + c_2 \cdot t$.

$X(0) = c_1 = 0$.

$X'(x) = c_2$

$X'(4) = c_2 = 0$

So $\lambda = 0$ is not an eigenvalue.

$\lambda > 0$. $\lambda = K^2$. The general solution is $X = c_1 \cdot \cos(K \cdot x) + c_2 \cdot \sin(K \cdot x)$.

$X(0) = c_1 \cdot \cos(K \cdot 0) = c_1 = 0$

$X'(x) = -c_1 \cdot K \cdot \sin(K \cdot x) + c_2 \cdot K \cdot \cos(K \cdot x) = c_2 \cdot K \cdot \cos(K \cdot x)$ because $c_1 = 0$.

$X'(4) = c_2 \cdot K \cdot \cos(4 \cdot K) = 0$. This can be true without $c_2 = 0$, if $\cos(4 \cdot K) = 0$. This happens if $4 \cdot K = \dfrac{(2n-1) \cdot \pi}{2}$, $n = 1, 2, \cdots$

or equivalently, $K = \dfrac{(2n-1) \cdot \pi}{8}$. The eigenvalues are $\dfrac{(2n-1)^2 \cdot \pi^2}{64}$ and the eigenfunctions are $X_n(x) = \sin\left(\dfrac{(2n-1) \cdot \pi}{8} \cdot x\right)$,

$n = 1, 2, \cdots$

Question 5

5. a. Find the coefficients b_n in the Fourier Sine series $\sum_{n=1}^{\infty} b_n \cdot \sin(n \cdot \pi \cdot x)$ for the function $f(x)$ where

$$f(x) = \begin{cases} 0 & 0 \le x < 0.4 \\ -1 & 0.4 \le x < 0.6 \\ 0 & 0.6 \le x \le 1 \end{cases}$$

b. Write out the first 3 non-zero terms of this series.

c. Which graph below could be a graph of this series for $n = 15$? Justify your answer.

Solution

a.

restart;

$L := 1:$

$$f := \begin{cases} 0 & 0 \le x < 0.4 \\ -1 & 0.4 \le x < 0.6 \\ 0 & 0.6 \le x \le 1 \end{cases} :$$

$$b_n := \frac{2}{L} \cdot \int_0^L f \cdot \sin\left(\frac{n \cdot \pi \cdot x}{L}\right) \mathrm{d}x$$

$$b_n := -\frac{0.6366197724\,(\cos(1.256637061\,n) - \cos(1.884955592\,n))}{n} \qquad \textbf{(5.1.1)}$$

b.

$k := 3:$

$$g := \sum_{n=1}^{k} b_n \cdot \sin\left(\frac{n\,\pi\,t}{L}\right)$$

$$g := -0.3934526574 \sin(3.141592654\,t) - 2.228169203\; 10^{-10} \sin(6.283185308\,t) + 0.3433574766 \sin(9.424777962\,t) \qquad \textbf{(5.1.2)}$$

c. The graph is the middle one because it's an odd function and approximates the function $f(x)$. The left graph is even, and the right one doesn't approximate f.