M344 - LAB 1: The Aging Spring

Due date: Wednesday, October 14, 2009

The behavior of an object of mass 1 kilogram, suspended on a spring which is decaying over time, can be modelled by the differential equation

$$x''(t) + bx'(t) + 3e^{-ct}x(t) = 0, \qquad (*)$$

where $x(t)$ is the position of the large object at time t. We will assume that the object is initially pulled down 1 meter ($x(0) = -1$) and given an initial velocity of 3 meters/second in the upward x-direction ($x'(0) = 3$).

If $c = 0$, the spring constant remains fixed at 3 Newtons per meter for all time. If $c > 0$, the restoring force of the spring is decreasing over time; that is, the spring is "aging". The constant b is used to simulate damping in the system.

Answer each of the following:

1. [**15 pts**] Let $c = 0$ so that the spring constant remains at 3 Newtons per meter for all t. Assume there is no damping ($b = 0$) and solve the equation $(*)$ using the characteristic polynomial method. Include a graph of the solution on the interval $0 \leq t \leq 50$. Find the period of the oscillatory solution and use it to determine how many times the object passes through its equilibrium position during this time interval.

2. [**10 pts**] Repeat the first problem with a small amount of damping $b = 0.2$. Include a graph of the solution on the interval $0 \leq t \leq 50$. Does the damping have a very big effect on the behavior of the system? Explain. What is the value of critical damping when $c = 0$?

3. [**10 pts**] Now let $c = 0.2$ so that the restoring force of the spring is continually decreasing (after 50 seconds it will have decreased to $3e^{-0.2(50)} \approx 0.00014$). Assume zero damping ($b = 0$) and use DEplot in MAPLE to plot a graph of the numerical solution $x(t)$ of $(*)$ on $0 \leq t \leq 50$.

4. [**20 pts**] Convert the differential equation $x'' + 3e^{-0.2t}x = 0$ to Bessel's equation of order 0 (see Lecture #8). Note that in equation (*) the spring constant $k = 3$. In the general solution

$$x(t) = C_1 J_0\left(\frac{2}{c}\sqrt{\frac{k}{m}}e^{-ct/2}\right) + C_2 Y_0\left(\frac{2}{c}\sqrt{\frac{k}{m}}e^{-ct/2}\right)$$

use the initial conditions $x(0) = -1, \quad x'(0) = 3$ to determine C_1 and C_2. Use MAPLE to plot the resulting function $x(t)$ on the interval $0 \le t \le 50$. It should look exactly like the graph in problem 3.

5. [**15 pts**] In problem 4 the total number of times the object passes through its equilibrium position $x(t) = 0$, before the restoring force is gone, is approximately equal to the number of zeros of $J_0(z)$ between $z = 0$ and $z = \frac{2}{c}\sqrt{\frac{k}{m}}$. Plot the function $J_0(t)$ and determine this number. How does it compare to the number of zero-crossings found in problems 3 and 4?

6. [**5 pts**] Does the fact that the spring constant is decreasing cause the zero-crossings to occur closer together in time or further apart as t increases?

7. [**5 pts**] Describe a way in which you could use the sign of the constants C_1 and C_2 to determine whether $x(t)$ will finally tend up or down as $t \to \infty$. Check that your idea works by seeing what happened in Problem 4.

8. [**10 pts**] To see the effect of damping on a system with an aging spring, use DEplot in MAPLE to graph numerical solutions of

$$x''(t) + bx'(t) + 3e^{-0.2t}x(t) = 0, \ \ x(0) = -1, \ x'(0) = 3$$

for two or three values of the damping constant b (between 0.2 and 0.6). Describe what happens as the damping increases?

9. [**10 pts**] **Write a brief introduction** to the Lab, giving a general description of what you did. This should appear at the top of the first page after the title page. **The title page should contain the course number M344, the title of the Lab, your name, and the due date.**

REMEMBER that every graph must have a title, both axes must have a label, and at least one hash mark on each axis showing the units being used.