M344 - Lab #1 Due date: Mon., October 11, 2010

Mass-spring equation with non-constant coefficients

A car in a garage is lifted to a height of 1 meter. The mechanism used to lower the car can be modelled by the initial value problem

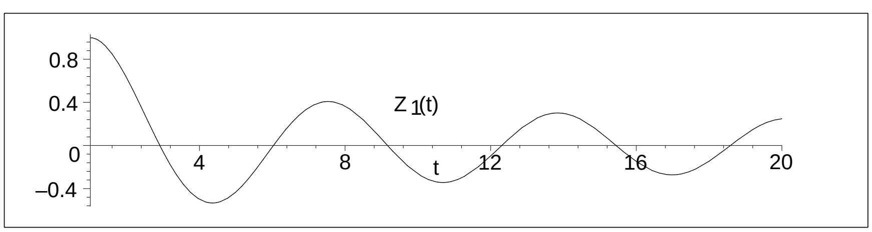
$$y'' + \frac{\beta}{t}y' + y = 0, \quad y(1) = 1.0, \quad y'(1) = 0, \tag{1}$$

where $y(t)$ is the height of the car at time $t \geq 1$. Note that the damping in the system is proportional to $\frac{1}{t}$, with constant of proportionality β. **Your job is to determine the value of β that gets the height of the car back to zero as quickly as possible, without ever letting it go below a height of -0.40 meters.**

Let $Z_\beta(t)$ denote the solution of equation (1) for any value of β.

1. If $\beta = 0$, equation (1) becomes the undamped mass-spring d.e. $y'' + y = 0$, with general solution $Z_0(t) = A_0 \cos(t) + B_0 \sin(t)$.
   - Find the constants A_0 and B_0 to make $Z_0(1) = 1, \; Z_0'(1) = 0$.
   - Draw a graph of $Z_0(t)$ for $1 \leq t \leq 20$.
   - How far below zero does $Z_0(t)$ go for $1 \leq t \leq \infty$?

2. If $\beta = 1$, equation (1) is Bessel's equation of order 0. We know that the general solution is $Z_1(t) = A_1 J_0(t) + B_1 Y_0(t)$.
   - Find the constants A_1 and B_1 to make $Z_1(1) = 1, \; Z_1'(1) = 0$. Remember that $J_0' = -J_1$ and $Y_0' = -Y_1$. You will need the four values
     $J_0(1) \approx 0.76519769, J_1(1) \approx 0.44005059, Y_0(1) \approx 0.088256964, Y_1(1) \approx -0.78121282.$
   - How far below zero does $Z_1(t)$ go for $1 \leq t \leq \infty$? Either use the graph below, or do the extra credit problem.



   - **Extra credit:** use MAPLE to draw a graph of $Z_1(t)$ for $1 \leq t \leq 20$.

3. If $\beta = 2$,
   - **Show** that $Z_2(t) = A_2\frac{\sin(t)}{t} + B_2\frac{\cos(t)}{t}$ satisfies equation (1).
   - Find the constants A_2 and B_2 to make $Z_2(1) = 1, \; Z_2'(1) = 0$.
   - Draw a graph of $Z_2(t)$ for $1 \leq t \leq 20$.
   - How far below zero does $Z_2(t)$ go for $1 \leq t \leq \infty$?

4. You are now going to use series to solve equation (1) for any $\beta \geq 0$.

- **Show** that $t = 0$ is a regular singular point of equation (1).
- **Show** that the roots of the indicial equation are 0 and $1 - \beta$.
- For the root $r = 0$, let $y_1(t) = \sum_{n=0}^{\infty} a_n t^n$ and **find a recursion formula** for the coefficients a_n.
- **Show** that $y_1(t) = a_0 P_\beta(t)$, where

$$P_\beta(t) = 1 - \frac{t^2}{2(1+\beta)} + \frac{t^4}{2 \cdot 4(1+\beta)(3+\beta)} - \cdots . \tag{2}$$

- Write out the first 6 non-zero terms in $P_\beta(t)$ and $P'_\beta(t)$.
- For the root $r = 1 - \beta$, let $y_2(t) = \sum_{n=0}^{\infty} b_n t^{n+1-\beta}$ and **find a recursion formula** for the coefficients b_n.
- **Show** that $y_2(t) = b_0 t^{1-\beta} Q_\beta(t)$, where

$$Q_\beta(t) = 1 - \frac{t^2}{2(3-\beta)} + \frac{t^4}{2 \cdot 4(3-\beta)(5-\beta)} - \cdots . \tag{3}$$

- Write out the first 6 non-zero terms in $Q_\beta(t)$ and $Q'_\beta(t)$.

You now have a general solution for equation (1) of the form

$$Z_\beta(t) = A_\beta P_\beta(t) + B_\beta t^{1-\beta} Q_\beta(t).$$

5. From what you know so far, **estimate** the **smallest** value of β such that $Z_\beta(t)$ remains above -0.4 for all $t \geq 1$. Call this value β^*; it should be your answer to the problem stated in the introduction.

For your chosen value β^*:

- Use the 6-term series for $P_\beta, P'_\beta, Q_\beta, Q'_\beta$ to compute $P_{\beta^*}(1), P'_{\beta^*}(1), Q_{\beta^*}(1)$, and $Q'_{\beta^*}(1)$.
- Find the constants A_{β^*} and B_{β^*} to make $Z_{\beta^*}(1) = 1$ and $Z'_{\beta^*}(1) = 0$.
- Draw a graph of $Z_{\beta^*}(t)$ for $1 \leq t \leq 5$. Use a computer graph method and keep 6 terms in each series.
- How far below zero does $Z_{\beta^*}(t)$ go?
- At what time t does $Z_{\beta^*}(t)$ first return to zero?

EXTRA CREDIT:

(i) Show that if $\beta = 1$, the series $P_1(t)$ is identical to the series for $J_0(t)$.

(ii) For $\beta = 2$, show that the series $P_2(t)$ is the Maclaurin series for $\sin(t)/t$.

(iii) Explain why $A_1 P_1(t) + B_1 Q_1(t)$ is NOT the GENERAL solution for $\beta = 1$.

Put a cover on your lab, a short introduction on the first inside page, and write only on one side of each page