1. **(20pts)** You have been given the job of solving a differential equation in which one of the non-constant coefficients is \( \sin(x) \). **Find the first 3 non-zero terms in the Taylor Series about \( x = 0 \) for the function \( f(x) = \frac{\sin(x)}{1-x} \).**

Hint: Either multiply the series for \( \sin(x) \) by the series for \( \frac{1}{1-x} \), or use the formula
\[
f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2} + \cdots.
\]

Answer:
\[
\sin(x) \frac{1}{1-x} = (x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots)(1 + x + x^2 + \cdots) = x + x^2 + \frac{2}{3}x^3 + \cdots
\]

2. **(16pts)** For each of the series below, use the Ratio Test to find its radius of convergence:

a) \( f(x) = \sum_{n=0}^{\infty} \frac{n+1}{2^n} x^n \)

b) \( g(x) = \sum_{n=1}^{\infty} \frac{n^2}{n!} \left(\frac{x}{2}\right)^n \)

Answer: (a) \( a_n = \frac{n+1}{2^n} \)

\[
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \frac{n + 2}{2^{n+1}} \frac{2^n}{n + 1} = \lim_{n \to \infty} \frac{n + 2}{2(n + 1)} = \frac{1}{2} = L.
\]

Therefore, the radius of convergence is \( R = 2 \).

(b) \( a_n = \frac{n^2}{n!} \)

\[
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \frac{(n + 1)^2}{(n + 1)!2^{n+1}} \frac{n!2^n}{n^2} = \lim_{n \to \infty} \frac{(n + 1)^2}{n^2} \frac{1}{2(n + 1)} = \lim_{n \to \infty} \frac{n + 1}{2n^2} = 0 = L.
\]

Therefore, the series converges for all \( x \).

3. **(20pts)** Solve the Cauchy-Euler initial-value problem:
\[
2x^2 y'' + 7xy' - 3y = 0, \quad y(1) = 1, \quad y'(1) = 4.
\]

Sketch a completely labelled graph of your solution on the interval \( 0 < x \leq 4 \).

Answer: \( y = 2x^{\frac{3}{2}} - x^{-3} \)
4. **(24pts)** For each differential equation below determine whether \( x = 0 \) is an ordinary point or a singular point. If it is a singular point, determine whether or not it is a regular singular point (show your computation of \( p_0 \) and \( q_0 \)).

(i) \( x^2y'' + 2x^2y' + 2(x^2 - 1)y = 0 \)

(ii) \( (1 - x^2)y'' + 2y' + x^2y = 0 \)

(iii) \( x^2y'' + (2 + x^2)y' + \frac{\sin(x)}{x}y = 0 \).

**Answer:**

(i) \( x = 0 \) is a **regular singular point**

(ii) \( x = 0 \) is an **ordinary point**

(iii) \( x = 0 \) is an **irregular singular point** since \( \lim_{x \to 0} x p(x) = \lim_{x \to 0} x(2 + x^2) = \infty \).

5. **(20pts)** Consider the differential equation \( xy'' + (1 - x)y' + 4y = 0 \). This is Laguerre’s equation of order 4.

(a) Show that \( r = 0 \) is a double root of the indicial equation.

(b) Substituting the series \( y = \sum_{n=0}^{\infty} a_n x^n \) and its derivatives into the equation gives

\[
\sum_{n=0}^{\infty} n(n-1)a_n x^{n-1} + \sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} m a_n x^n + \sum_{n=0}^{\infty} 4a_n x^n \equiv 0.
\]

Change two summation indices and find a **recurrence formula** for the \( a_n \). You can assume \( a_0 \neq 0 \).

**Extra Credit:** Show that the series solution you get is a 4th degree polynomial in \( x \), and find its coefficients.

**Answer:**

(a) \( p_0 = \lim_{x \to 0} x \frac{1-x}{x} = 1, q_0 = \lim_{x \to 0} x^2 \frac{4}{x} = 0 \)

The indicial equation \( r^2 + (p_0 - 1)r + q_0 = r^2 = 0 \) has roots \( r = 0, 0 \).

(b) Letting \( m = n - 1 \) in the first two sums,

\[
\sum_{m=0}^{\infty} (m+1)a_{m+1} x^m + \sum_{m=0}^{\infty} m a_n x^n - \sum_{m=0}^{\infty} m a_m x^m + \sum_{m=0}^{\infty} 4a_m x^n \equiv 0
\]

The recursion formula is \( a_{m+1} = \frac{(m-1)a_m}{(m+1)x^2} \).

Ex. Credit: Let \( a_0 \) be arbitrary. Then \( a_1 = -4a_0, a_2 = -3a_1/2^2 = 3a_0, a_3 = -2a_2/3^2, a_4 = -a_3/4^2 \) and \( a_5 = a_6 = \cdots = 0 \). The solution is the polynomial \( y(t) = a_0(1 - 4t + 3t^2 - \frac{2}{3}t^3 + \frac{1}{24}t^4) \).