M344 - EXAM # 1 with Answers

1. **(20pts)** You have been given the job of solving a differential equation in which one of the non-constant coefficients is $\frac{\sin(x)}{1-x}$. **Find the first 3 non-zero terms in the Taylor Series about $x = 0$ for the function $f(x) = \frac{\sin(x)}{1-x}$.**

   Hint: Either multiply the series for $\sin(x)$ by the series for $\frac{1}{1-x}$, or use the formula $f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots$.

   Answer:
$$\frac{\sin(x)}{1-x} = (x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots)(1 + x + x^2 + x^3 + \cdots) = x + x^2 + \frac{2}{3}x^3 + \cdots$$

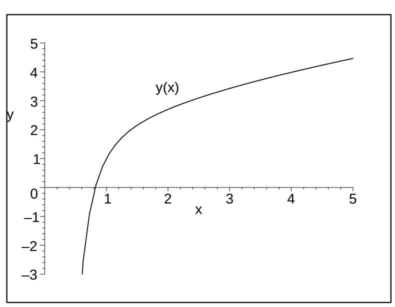
2. **(16pts)** For each of the series below, use the Ratio Test to find its radius of convergence:

   a) $f(x) = \sum_{n=0}^{\infty} \frac{n+1}{2^n} x^n$ b) $g(x) = \sum_{n=1}^{\infty} \frac{n^2}{n!} \left(\frac{x}{2}\right)^n$

   Answer: (a) $a_n = \frac{n+1}{2^n}$
$$\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \lim_{n\to\infty} |\frac{n+2}{2^{n+1}} \frac{2^n}{n+1}| = \lim_{n\to\infty} |\frac{n+2}{2(n+1)}| = \frac{1}{2} = L.$$
   Therefore, the radius of convergence is $R = 2$.

   (b) $a_n = \frac{n^2}{n!2^n}$
$$\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \lim_{n\to\infty} |\frac{(n+1)^2}{(n+1)!2^{n+1}} \frac{n!2^n}{n^2}| = \lim_{n\to\infty} |\frac{(n+1)^2}{n^2} \frac{1}{2(n+1)}| = \lim_{n\to\infty} |\frac{n+1}{2n^2}| = 0 = L.$$
   Therefore, the series converges for all x.

3. **(20pts)** Solve the Cauchy-Euler initial-value problem:
$$2x^2y'' + 7xy' - 3y = 0, \quad y(1) = 1, \quad y'(1) = 4.$$
   Sketch a completely labelled graph of your solution on the interval $0 < x \leq 4$.



   Answer: $y = 2x^{\frac{1}{2}} - x^{-3}$

4. **(24pts)** For each differential equation below determine whether $x = 0$ is an ordinary point or a singular point. If it is a singular point, determine whether or not it is a regular singular point (show your computation of p_0 and q_0).

(i) $x^2y'' + 2x^2y' + 2(x^2 - 1)y = 0$

(ii) $(1 - x^2)y'' + 2y' + x^2y = 0$

(iii) $x^2y'' + (2 + x^2)y' + \frac{\sin(x)}{1-x}y = 0.$

Answer: (i) $x = 0$ is a **regular singular point**

(ii) $x = 0$ is an **ordinary point**

(iii) $x = 0$ is an **irregular singular point** since $\lim_{x\to 0} xp(x) = \lim_{x\to 0} \frac{x(2+x^2)}{x^2} = \infty$.

5. **(20pts)** Consider the differential equation $xy'' + (1-x)y' + 4y = 0$. This is Laguerre's equation of order 4.

(a) Show that $r = 0$ is a double root of the indicial equation.

(b) Substituting the series $y = \sum_{n=0}^{\infty} a_nx^n$ and its derivatives into the equation gives

$$\sum_{n=0}^{\infty} n(n-1)a_nx^{n-1} + \sum_{n=1}^{\infty} na_nx^{n-1} - \sum_{n=0}^{\infty} na_nx^n + \sum_{n=0}^{\infty} 4a_nx^n \equiv 0.$$

Change two summation indices and find a recurrence formula for the a_n. You can assume $a_0 \neq 0$.

Extra Credit: Show that the series solution you get is a 4th degree polynomial in x, and find its coefficients.

Answer: (a) $p_0 = \lim_{x\to 0} x(\frac{1-x}{x}) = 1, q_0 = \lim_{x\to 0} x^2\frac{4}{x} = 0$

The indicial equation $r^2 + (p_0 - 1)r + q_0 = r^2 = 0$ has roots $r = 0, 0$.

(b) Letting $m = n - 1$ in the first two sums,

$\sum_{m=0}^{\infty}(m+1)ma_{m+1}x^m + \sum_{m=0}^{\infty}(m+1)a_{m+1}x^m - \sum_{m=0}^{\infty} ma_mx^m + \sum_{m=0}^{\infty} 4a_mx^m \equiv 0$

The recursion formula is $a_{m+1} = \frac{(m-4)a_m}{(m+1)^2}$.

Ex. Credit: Let a_0 be arbitrary. Then $a_1 = -4a_0, a_2 = -3a_1/2^2 = 3a_0, a_3 = -2a_2/3^2, a_4 = -1a_3/4^2$ and $a_5 = a_6 = \cdots = 0$. The solution is the polynomial $y(t) = a_0(1 - 4t + 3t^2 - \frac{2}{3}t^3 + \frac{1}{24}t^4)$.