Non Homogeneous Equations - Resonance
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Initial equation to solve: \[ m \frac{d^2 x}{dt^2} + c \frac{dx}{dt} + kx = F(t) \] (1)

Divide by \( m \):
\[ \frac{d^2 x}{dt^2} + \frac{c}{m} \frac{dx}{dt} + \frac{k}{m} x = \frac{1}{m} F(t) \] (2)

Recast as
\[ \frac{d^2 x}{dt^2} + 2\omega_0 \xi \frac{dx}{dt} + \omega_0^2 x = \frac{1}{m} F(t) \] (3)

Where:
\[ \xi = \frac{1}{2\omega_0} \frac{c}{m} \quad \text{and} \quad \omega_0 = \sqrt{\frac{k}{m}} \] (4)

First solve the homogeneous equation to get the complementary solution (solutions to the homogeneous equation),
\[ \frac{d^2 x}{dt^2} + 2\omega_0 \xi \frac{dx}{dt} + \omega_0^2 x = 0 \] (5)

Then use the solutions to find the particular solution, \( x_p \).

**Solution to nonhomogeneous equation, the particular solution, \( x_p \)**

Assume \( F(t) = F_0 \sin(\omega t) \)
\[ \frac{d^2 x_p}{dt^2} + 2\omega_0 \xi \frac{dx_p}{dt} + \omega_0^2 x_p = \frac{F_0}{m} \sin(\omega t) \] (7)

\[ x_p(t) = b_1 \sin \omega t + b_1 \cos \omega t = A \sin(\omega t + \phi) \] (8)

Manipulation of this equation gives (recalling that \( \omega_0^2 = \frac{k}{m} \))
\[ A = \frac{F_0}{k} \left[ \frac{1}{\sqrt{1 - \left( \frac{\omega}{\omega_0} \right)^2}} + \left( \frac{2 \xi}{\omega_0} \right)^2 \right] \quad \tan \phi = 2\xi \frac{\omega}{\omega_0} \left( \frac{\omega}{\omega_0} \right)^2 - 1 \] (9a,b)

Define the relative frequency as \( \omega_r = \frac{\omega}{\omega_0} \). Then equations (9) become
\[ A = \frac{F_0}{k} \left[ \frac{1}{\sqrt{1 - \omega_r^2}} + (2\xi \omega_0)^2 \right] \quad \tan \phi = 2\xi \frac{\omega_r}{\omega_r^2 - 1} \] (10a,b)
Note that, from the graph below, that the maximum value of the amplitude is near $\omega_r = 1$ (for small values of $\xi$). The actual value is found by taking the derivative of the amplitude then setting it equal to zero. The resulting location of the maximum is found to be

$$\omega_r = \sqrt{1 - 2\xi^2} \quad \text{with} \quad \xi^2 \leq \frac{1}{2} = \left(0.7071\right)^2$$  \hspace{1cm} (11)

Using this value of $\omega_r$, the maximum amplitude is found to be

$$A = \frac{F_0}{2k} \frac{1}{\sqrt{\xi^2 \left[1 - \xi^2\right]}}$$  \hspace{1cm} (12)

In the following graphs, the letter $z$ in the legend represents $\xi$.

![Resonance Graph](image)

The value of $\frac{F_0}{k}$ in the above graph was taken to be unity: $\frac{F_0}{k} = 1$. The strength, $F_o$, of the applied force used for the graph only affects the amplitude. The frequency of oscillation depends on the natural frequency, $\omega_0$, and the frequency of the driving force, $\omega$. Hence, care must be taken when applying sinusoidal forces for small values of damping ratio, $\xi$. The less frictional (resistance) loss there is to a system the more the resonance is fed. For the case of no friction (LC circuits), the amplitude is infinite when $\omega_r = 1$, i.e. the system is being fed energy at its natural frequency and there is no loss of energy.
The phase values look like the typical arctan curves. When $\omega_r = 1$, the denominator vanishes so the angle is $\pm90^\circ$.

**Full solution to nonhomogeneous equation**

The full solution to the problem is given by

$$x(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} + x_p(t)$$

(13)

The solution has a transient part, the complementary solution $x_c(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$, and the steady state part, $x_p(t)$. With no forcing function a damped harmonic oscillation will decay to no motion. The sinusoidal forcing function keeps the system oscillating by supplying energy to the system to counteract frictional losses.
RLC Circuits

The mechanical equation we are solving is

\[ m \frac{d^2x}{dt^2} + c \frac{dx}{dt} + kx = F_0 \sin \omega t \]  \hspace{1cm} (14)

The series RLC circuit equation to solve is given by the Kirchhoff voltage law:

\[ L \frac{d^2Q}{dt^2} + R \frac{dQ}{dt} + \frac{1}{C}Q = V_0 \sin \omega t \]  \hspace{1cm} (15)

By appropriate identification of parameters, all the above analysis holds for RLC series circuits.

\[ m \leftrightarrow L \quad c \leftrightarrow R \quad k \leftrightarrow \frac{1}{C} \quad F_0 \leftrightarrow V_0 \]  \hspace{1cm} (16)

Similar results hold for parallel RLC circuits using Kirchhoff’s nodal analysis. See a previous handout for more information.