Newton’s Law of Cooling

Our text writes that Newton’s Law of Cooling obeys the differential equation
\[ \frac{dT}{dt} = -k(T - T_m) \]
where \( T \) is the time-dependent temperature of the object, \( T_m \) is the temperature of the medium which contains the object, and \( k \) is a materials dependent constant.

This equation can be solved in many ways.

1) First, we look at the equation as a separable equation. Change the variable from just \( T \) to \( T - T_m \). Then the equation can be written as
\[ \frac{d(T - T_m)}{dt} = -k(T - T_m) \]
This form leads to
\[ \frac{d(T - T_m)}{(T - T_m)} = -k dt \]
The solution is
\[ T - T_m = Ce^{-kt} \]
i.e.
\[ T = T_m + Ce^{-kt} \]

2) Now look at the equation as a linear equation.
\[ \frac{dT}{dt} + kT = kT_m \]
Using the formula for the solution of a linear equation gives
\[ I(t) = e^{kt} \]
\[ \text{Integrating factor} \]
\[ T(t) = \frac{1}{e^{kt}} \left[ C + \int e^{kt} kT_m \, dt \right] \]
Since \( T_m \) and \( k \) are constants, they pull out of the integral giving
\[ T(t) = \frac{1}{e^{kt}} \left[ C + kT_m \int e^{kt} \, dt \right] = \frac{1}{e^{kt}} \left[ C + kT_m \frac{e^{kt}}{k} \right] \]
A little algebra gives
\[ T(t) = \frac{1}{e^{kt}} \left[ C + T_m e^{kt} \right] = T_m + \frac{C}{e^{kt}} = T_m + Ce^{-kt} \]
The general solution is then given by
\[ T = T_m + Ce^{-kt} \]

Next we give the constants \( C \) and \( k \) physical meaning.
3) We apply the initial condition \( T(0) = T_0 \) so
\[
T_0 = T_m + C
\]
Hence
\[
C = T_0 - T_m
\]
The solution is therefore
\[
T(t) = T_m + (T_0 - T_m)e^{-kt}
\]
Note that when \( t=0 \), the solution gives
\[
T(t) = T_m + (T_0 - T_m) = T_0
\]
and for long times,
\[
T(t) \to T_m
\]
as one would expect.

How long is a long time? EETs say that \( e^{-5} \) is close enough to 0, so when \( t = \frac{k}{5} \), the long time condition is reached. The constant \( k \) can now be given physical meaning. It is the inverse of the time constant, \( \tau \), for the problem.
\[
k = \frac{1}{\tau}
\]
A physically meaningful way of writing the solution of Newton’s Law of Cooling is therefore
\[
T(t) = T_m + (T_0 - T_m)e^{-\frac{t}{\tau}}
\]
Every symbol has a physical meaning.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
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<tbody>
<tr>
<td>( t )</td>
<td>The time</td>
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<tr>
<td>( T(t) )</td>
<td>The temperature of the object as a function of time.</td>
</tr>
<tr>
<td>( T_m )</td>
<td>The temperature of the medium.</td>
</tr>
<tr>
<td>( T_0 )</td>
<td>The initial temperature of the object.</td>
</tr>
<tr>
<td>( \tau )</td>
<td>The time constant of the object in the medium. When ( t &gt; 5\tau ), the temperature of the object has essentially reached the temperature of the medium.</td>
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