Sections covered:
24.1 Tangents and Normals
24.3 Curvilinear Motion
24.4 Related Rates
24.5 Using Derivatives in Curve Sketching
24.7 Applied Maximum and Minimum Problems

Note that in the examples below, you don't see much algebra or derivative detail. On the exam, you are free to use your calculator for those calculations. Follow the details of the solutions below to show me your work. For example, state the derivative you are taking, given
\[ y = -\frac{1}{7}x + \frac{51}{7}, \quad \frac{dy}{dx} = -\frac{1}{7}. \] That is all you have to do to show your work. For algebra, say something like "using Solver:" or "using Factor:" or "using Expand:". Note – don't plug in values of variables until the derivatives are taken. All problems involve differentiation.

24.1 Tangents and Normals

Notation: a given point will be denoted as \((x_1, y_1)\)

As developed in section 23.2, the slope of the tangent line at a point is the first derivative of the function evaluated at that point.

\[ m = \left. \frac{dy}{dx} \right|_{x=x_1} \]

Since we will be dealing with both tangents and normal, I will use subscripts to distinguish between them when necessary.

\[ m_{\text{tan}} = \left. \frac{dy}{dx} \right|_{x=x_1} \quad \text{and} \quad m_\perp = -\frac{1}{m_{\text{tan}}}. \]

We solved four types of problems in this section.
A) Find the equation of the line tangent to the given curve, given the point \((x_1, y_1)\).
B) Find the equation of the line normal to the given curve, given the point \((x_1, y_1)\).
C) Find the equation of the line tangent to the given curve, given the slope \(m\).
D) Find the equation of the line normal to the given curve, given the slope \(m\).

Following are examples of each.
A) Find the equation of the line tangent to the curve \( y = 2x^2 - x + 1 \) at the point \( x = 2 \).

1) Take the derivative of the function. \[
\frac{dy}{dx} = 4x - 1
\]

2) Evaluate the slope. \[
m = \frac{dy}{dx}\bigg|_{x=2} = 4(2) - 1 = 7
\]

3) Find the \( y \) value at the point. \[
y = 2(2)^2 - 2 + 1 = 7
\]

4) Find the value of the \( y \) intercept. \[
b = y_1 - mx_1 = 7 - 7 \cdot 2 = -7
\]

5) Write the equation of the tangent line. \[
y = 7x - 7
\]

Check by graphing both the original function, \( y = 2x^2 - x + 1 \), and the equation of the tangent line, \( y = 3x + 1 \).

1) Define the functions (◆F1).

2) Window – make sure the point of interest is within the window of the graph. I also like to make sure the two axes show. Note that I first graphed the function using ZoomStd to see what it looks like. Finally, I chose the following window after playing with some of the values.

3) The graph.
If instead of graphing the tangent line as a second equation, I just use F5/A when looking at the graph, I get the following, thus verifying my tangent line equation.

B) Find the equation of the line normal to the curve $y = 2x^2 - x + 1$ at the point $x = 2$.

1) We know the tangent slope from the above. $m_{\text{tan}} = \frac{dy}{dx}_{x=2} = 7$

Here we want the normal slope. $m_{\perp} = -\frac{1}{m_{\text{tan}}} = -\frac{1}{7}$

2) Find the value of the $y$ intercept. $b = y_i - mx_i = 7 - \left(\frac{-1}{7}\right) \cdot 2 = \frac{51}{7}$

3) Write the equation of the normal line. $y = -\frac{1}{7}x + \frac{51}{7}$

4) Check by graphing both the original function, $y = 2x^2 - x + 1$, and the equation of the normal line, $y = -\frac{1}{7}x + \frac{51}{7}$. Since the original function and the point are the same as above, I will just include the normal line equation, then graph both.

The graph

The line does not look perpendicular (normal). So, F2/ZoomSqr.
Now I will add the tangent line (F5/A) to make sure the line is perpendicular. For this example, the above graph looks perpendicular but for some graphs it will not be so obvious.

C) Find the equation of the line tangent to the curve \( y = 2x^2 - x + 1 \) given the slope, \( m = 7 \).

You are given the slope so you need to find the point on the curve where the slope is 7.
1) Take the derivative of the function. \( \frac{dy}{dx} = 4x - 1 \)
2) Find the value of \( x \).
\[
\frac{dy}{dx} = 4(x) - 1 = m = 7 \rightarrow x = \frac{8}{4} = 2
\]
3) You know both \( m \) and \( x \) so follow example A above to finish the problem.

D) Find the equation of the line normal to the curve \( y = 2x^2 - x + 1 \) given the slope, \( m = -\frac{1}{7} \).

1) Find the tangent line slope so you can equate it to the derivative to find \( x \).
\[
m_{\text{tan}} = -\frac{1}{m_{\perp}} = -1/(-1/7) = 7
\]
2) Find the value of \( x \) given the tangent slope. \( \frac{dy}{dx} = 4(x) - 1 = m = 7 \rightarrow x = \frac{8}{4} = 2
\)
3) You know both \( m \) and \( x \) so follow example B above to finish the problem.
24.3 Curvilinear Motion

In PHY 120 you learned that

\[ x = x_0 + v_0 t + \frac{1}{2} a t^2 \quad \text{and} \quad v = v_0 + a t \]

A more general form is given an equation for \( x \), \( v_x = \frac{dx}{dt} \) and given an equation for \( y \), \( v_y = \frac{dy}{dt} \).

The acceleration is no longer constant, it is the derivative of the velocity.

\[ a_x = \frac{dv_x}{dt} \quad \text{and} \quad a_y = \frac{dv_y}{dt} \]

In class we solved two forms of problems.

A) Given \( x \) and \( y \) as functions of time, i.e. \( x(t) \) and \( y(t) \), find the velocity and acceleration.

B) Given \( y \) as functions of \( x \) and \( v_x \), find the velocity.

Examples:

A) Find the velocity and acceleration given \( x = \frac{1}{t+1} \) and \( y = \sqrt{t^3 + 1} \) at \( t = 2 \).

1) Straight TI-89 solution - velocity.

a) Find the derivative

\[ \frac{d}{dt} \left( \frac{1}{t+1}, \sqrt{t^3 + 1} \right) \]

\[ \frac{-1}{(t+1)^2} \frac{3t^2}{2 \sqrt{t^3 + 1}} \]

The notation is \( \vec{v} = d([x,y], t) \). Note that the TI removes the comma from the vector brackets when it shows the result.

b) Plug in the time

\[ \left[ \frac{-1}{(t+1)^2} \frac{3t^2}{2 \sqrt{t^3 + 1}} \right] t = 2 \]

\[ \begin{bmatrix} -\frac{1}{9} \ 2 \ \\
-\frac{1}{9} \ 2 \ \\
\end{bmatrix} \]

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c) Convert the answer to polar coordinates.

You can see from the bottom of the screen that the angle mode is DEG i.e. degrees.

2) Straight TI-89 solution - acceleration.

a) Find the derivative of the velocity found above or enter the following

The notation is \( \ddot{a} = d(\begin{bmatrix} v_x, v_y \end{bmatrix}, t) = d(\begin{bmatrix} x, y \end{bmatrix}, t, 2) \).

b) Plug in the time

c) Convert the answer to polar coordinates.
1) By hand solution - velocity.

a) Find the derivative

\[ v_x = \frac{dx}{dt} = \frac{d}{dt} \left( \frac{1}{t+1} \right) = \frac{d}{dt} (t+1)^{-1} = -(t+1)^{-2} \]

Note that \( \frac{d}{dt} (t+1) = 1 \) so the third factor in the power rule, \( \frac{d}{dx} u^n = nu^{n-1} \frac{du}{dx} \), was 1.

\[ v_y = \frac{dy}{dt} = \frac{d}{dt} \left( \sqrt{t^3+1} \right) = \frac{d}{dt} \left( (t^3+1)^{1/2} \right) = \frac{1}{2} (t^3+1)^{-1/2} \left( 3t^2 \right) = \frac{3t^2}{2\sqrt{t^3+1}} \]

b) Plug in the time

\[ v_x = -(2+1)^{-2} = \frac{1}{3^2} = \frac{1}{9} \]

\[ v_y = \frac{3(2)^2}{2\sqrt{2^3+1}} = \frac{3 \cdot 2}{\sqrt{9}} = 2 \]

c) Convert the answer to polar coordinates.

\[ v = \sqrt{v_x^2 + v_y^2} = \sqrt{\left( -\frac{1}{9} \right) + 2^2} = 2.003 \]

\[ \tan(\theta_v) = \frac{v_y}{v_x} \]

\[ \theta_v = \tan^{-1} \left( \frac{v_y}{v_x} \right) = \tan^{-1} \left( \frac{2}{-1/9} \right) = \tan^{-1} (-18) = -86.82^\circ \]

But the velocity is in the second quadrant since the \( x \) component is negative and the \( y \) component is positive. Whenever the \( x \) component is negative, add 180° to the calculator answer.

\[ \theta_v = -86.82^\circ + 180^\circ = 93.18^\circ \]

2) By hand solution - acceleration.

a) Find the derivative of the velocity found above.

\[ v_x = -(t+1)^{-2} \]

\[ a_x = \frac{dv_x}{dt} = -\left\{ -2(t+1)^{-3} \right\} = \frac{2}{(t+1)^3} \]

Note – I prefer the product rule to the quotient rule as \( u \) and \( v \) are interchangeable in the product rule and not in the quotient rule.

\[ v_y = \frac{3}{2} t^2 \left( t^3 + 1 \right)^{1/2} \]
\[ a_y = \frac{dv_y}{dt} = \frac{3}{2} \left\{ 2t(t^3 + 1)^{-1/2} + t^2 \left( -\frac{1}{2} \right)(t^3 + 1)^{-3/2} (3t^2) \right\} \]

\[ a_y = \frac{3}{2} \left( t^3 + 1 \right)^{-3/2} \left\{ 2t(t^3 + 1) + t^2 \left( -\frac{1}{2} \right)(3t^2) \right\} = \frac{3}{2(t^3 + 1)^{3/2}} \left( \frac{t^4}{2} + 2t \right) \]

\[ a_y = \frac{3t(t^3 + 4)}{4(t^3 + 1)^{3/2}} \]

b) Plug in the time, t=2.

\[ a_x = \frac{2}{(2+1)^3} = \frac{2}{27} \quad a_y = \frac{3 \cdot 2(2^3 + 4)}{4(2^3 + 1)^{3/2}} = \frac{3(12)}{2(9)^{3/2}} = \frac{3(12)}{2(27)} = \frac{2}{3} \]

\[ a_x = \sqrt{a_x^2 + a_y^2} = 0.671 \]

\[ \theta = \tan^{-1} \left( \frac{a_y}{a_x} \right) = \tan^{-1} \left( \frac{2/3}{2/27} \right) = \tan^{-1} (9) = 83.7^\circ \]

\[ a_x \text{ is positive so do not add } 180^\circ \text{ to the calculator answer.} \]
B) Given \( v_x \) and \( y \) as functions of \( x \), find the \( y \) component of the velocity.

Find the velocity given \( y = 2x^3 + x \) and \( v_x = 3 \) at \( t = 2 \).

Since, in general, we want the velocity, \( \vec{v} = \left[ \frac{dx}{dt}, \frac{dy}{dt} \right] \) but \( y \) is given as a function of \( x \), not \( t \), so we need to use the chain rule.

\[
v_y = \frac{dy}{dt} = \frac{dy}{dx} \frac{dx}{dt} = \frac{dy}{dx} v_x
\]

Solution.

\[
v_y = \frac{dy}{dx} v_x = \frac{d}{dx} (2x^3 + x) \cdot v_x = \left(6x^2 + 1\right) \cdot (3) = 18x^2 + 3
\]

Since \( v_x \) is constant, we know that \( x = v_x t = 3t \).

\[
v_y = 18(3t)^2 + 3 = 18(6)^2 + 3 = 651.
\]

Note, if \( v_x \) is not constant, we need integration (MTH 241) to find \( x \) as a function of time.

Once you have both components of velocity, you can find their polar representation by methods used above.

### 24.4 Related Rates

Every problem in related rates is essentially the same.

You are given one rate (either \( \frac{dx}{dt} \) or \( \frac{dy}{dt} \)) and the value of \( x \).

To find the other rate use the chain rule: \( \frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} \).

Plug in the value of \( x \) into the derivative \( \frac{dy}{dx} \).

For volume and surface area formulas, look in the front cover of your book. A couple of good examples you might want to download are found at [http://www.austinec.edu/pintutor/pin_mh/ source/Handouts/Geometry_Formulas/Geometry_Formulas_2D_3D_Perimeter_Area_Volume.pdf](http://www.austinec.edu/pintutor/pin_mh/ source/Handouts/Geometry_Formulas/Geometry_Formulas_2D_3D_Perimeter_Area_Volume.pdf) and [http://www.science.co.il/Formula.asp](http://www.science.co.il/Formula.asp)

In the following examples, I will switch the notation to \( x \) and \( y \) alongside using the actual variables so you can see the pattern better.

The examples below are from pages 730-733.
14. The voltage \( V \) that produces a current \( I \) (in A) in a wire of radius \( r \) (in in.) is \( V = \frac{0.030}{r^2} \). If the current increases at 0.020 A/s in a wire of 0.040 in. radius, find the rate at which the voltage is increasing.

\[
\begin{array}{|c|c|}
\hline
\text{Actual Problem} & \text{\( x \) and \( y \) version} \\
\hline
V = 0.030 \frac{I}{r^2} = \frac{0.030}{r^2} I & y = 0.030 \frac{x}{r^2} = \frac{0.030}{r^2} x \\
\frac{dl}{dt} = 0.020 \frac{A}{s} & \frac{dx}{dt} = 0.020 \quad r = 0.040 \text{in} \\
r = 0.040 \text{in} & \\
\hline
\frac{dV}{dt} = \frac{dV}{dr} \frac{dr}{dt} & \frac{dy}{dt} = \frac{dy}{dx} \frac{dx}{dt} \\
\frac{dV}{dl} = \frac{0.030}{r^2} \frac{dl}{dt} & \frac{dy}{dx} = \frac{0.030}{r^2} \frac{dx}{dt} \\
\frac{dV}{dt} = \left( \frac{0.030}{r^2} \right) \frac{dl}{dt} = \left( \frac{0.030}{(0.040)^2} \right) (0.020) = 0.375 \frac{V}{s} & \frac{dy}{dt} = \left( \frac{0.030}{r^2} \right) \left( \frac{dx}{dt} \right) = \left( \frac{0.030}{(0.040)^2} \right) (0.020) = 0.375 \\
\hline
\end{array}
\]

22. An engine cylinder 15.0 cm deep is being bored such that the radius increases by 0.100 mm/min. How fast is the volume \( V \) of the cylinder changing when the diameter is 9.50 cm?

\[
\begin{array}{|c|c|}
\hline
\text{Actual Problem} & \text{\( x \) and \( y \) version} \\
\hline
V = \pi r^2 h & y = \pi x^2 h \\
\frac{dr}{dt} = 0.100 \frac{mm}{min} = 0.010 \frac{cm}{min} & \frac{dx}{dt} = 0.010 \frac{cm}{min} \\
r = 9.5 \text{ cm} = 4.75 \text{ cm} \quad h = 15.0 \text{ cm} & x = 4.75 \text{ cm} \quad h = 15.0 \text{ cm} \\
\hline
\end{array}
\]
\[
\frac{dV}{dt} = \frac{dV}{dr} \frac{dr}{dt} \quad \text{h is constant} \\
\frac{dV}{dr} = \pi h \frac{dr^2}{dr} = 2\pi hr \\
\frac{dV}{dt} = \frac{dV}{dr} \frac{dr}{dt} = (2\pi hr) \frac{dr}{dt} =
\]
\[
(2\pi \cdot 15 \cdot 4.75)(0.010) = 4.477 \text{ cm}^3/\text{min}
\]

24. A rectangular image 4.00 in. high on a computer screen is widening at the rate of 0.25 in./s. Find the rate at which the diagonal is increasing when the width is 6.50 in.

\[
D = \sqrt{w^2 + h^2} \quad \frac{dw}{dt} = 0.25 \text{ in/}s \\
h = 4.0 \text{ in}
\]

\[
\frac{dD}{dt} = \frac{dD}{dw} \frac{dw}{dt} \\
\frac{dD}{dw} = \frac{d}{dw} \sqrt{w^2 + h^2} = \frac{d}{dw} (w^2 + h^2)^{1/2} = \frac{1}{2}(w^2 + h^2)^{-1/2} (2w) \frac{dw}{dt} = \frac{2w}{2\sqrt{w^2 + h^2}} \frac{dw}{dt} \\
\frac{dD}{dt} = \frac{2w}{2\sqrt{w^2 + h^2}} \frac{dw}{dt} = \frac{2 \cdot 6.5}{2\sqrt{6.5^2 + 4^2}} (0.25) =
\]
\[
0.213 \text{ in/}s
\]
34. The speed of sound \( v \) (in m/s) is \( v = 331 \sqrt{T/273} \), where \( T \) is the temperature (in K). If the temperature is 303 K (30°C) and is rising at 2.0°C/h, how fast is the speed of sound rising?

<table>
<thead>
<tr>
<th>Actual Problem</th>
<th>( x ) and ( y ) version</th>
</tr>
</thead>
<tbody>
<tr>
<td>( v = 331 \sqrt{T/273} = \frac{331}{\sqrt{273}} \sqrt{T} )</td>
<td>( y = 331 \sqrt{x/273} = \frac{331}{\sqrt{273}} \sqrt{x} )</td>
</tr>
<tr>
<td>( \frac{dT}{dt} = 2.0 \frac{^\circ C}{h} = \frac{2.0 \frac{^\circ K}{h}}{h} ) since (^\circ K = ^\circ C + 273 )</td>
<td>( \frac{dx}{dt} = 2.0 )</td>
</tr>
</tbody>
</table>

\[
\frac{dy}{dt} = \frac{dy}{dx} \frac{dx}{dt} \]

\[
\frac{dy}{dt} = \frac{331}{\sqrt{273}} \left( \frac{1}{2} T^{-1/2} \right) = \frac{331}{2 \sqrt{T \sqrt{273}}} \]

\[
\frac{dy}{dt} = \left( \frac{331}{\sqrt{273} \sqrt{x}} \right) \left( \frac{dx}{dt} \right) = \frac{331}{2 \sqrt{x \sqrt{273}}} \]

\[
\frac{dy}{dt} = \left( \frac{331}{2 \sqrt{273 \cdot 303}} \right) \left( 2 \right) = 1.151 \]

\[
1.151 \frac{m}{s \cdot h} = 1.151 \frac{m}{s \cdot h} \left( \frac{1 h}{3600 s} \right) = 4.144 E 3 \frac{m}{s^2} \]

24.5 Using Derivatives in Curve Sketching

<table>
<thead>
<tr>
<th>( \frac{dy}{dx} )</th>
<th>( \frac{d^2y}{dx^2} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>&gt;0</td>
<td>&gt;0</td>
</tr>
<tr>
<td>=0</td>
<td>=0 or</td>
</tr>
<tr>
<td>&lt;0</td>
<td>&lt;0</td>
</tr>
</tbody>
</table>
24.7 Applied Maximum and Minimum Problems

There are two types of max/min problems we have met.
1) The first is \( y \) is a function of \( x \), find the max/min value of \( y \).
2) The second involves two equations. \( u \) is a function of \( x \) and \( y \), and \( v \) is a function of \( x \) and \( y \). Find the max/min of one of them given the value of the other one.

The examples below are from pages 730-733.

Type 1 Max/Min

The first max/min type is where \( y \) is a function of \( x \), find the max/min value of \( y \).

37. The deflection \( y \) of a beam of length \( L \) at a horizontal distance \( x \) from one end is given by \( y = k(2x^4 - 5Lx^3 + 3L^2x^2) \), where \( k \) is a constant. For what value of \( x \) does the maximum deflection occur?

i) The equation: \( y = k(2x^4 - 5Lx^3 + 3L^2x^2) \) where both \( k \) and \( L \) are constant.

ii) Find the two derivatives of \( y \).
\[
\frac{dy}{dx} = k\left(8x^3 - 15Lx^2 + 6L^2x\right)
\]
\[
\frac{d^2y}{dx^2} = k\left(24x^2 - 30Lx + 6L^2\right)
\]

iii) Solve for the location of the minimum.
\[
\frac{dy}{dx} = k\left(8x^3 - 15Lx^2 + 6L^2x\right) = 0
\]
\[
x = 0 \quad \text{clearly not the answer physically.}
\]
\[
x = 1.3L \quad x \text{ cannot be larger than } L.
\]
\[
x = 0.578L \quad \text{must be the answer}
\]

iv) Check the second derivative.
\[
\frac{d^2y}{dx^2} = k\left(24(0.578L)^2 - 30L(0.578L) + 6L^2\right) = -3.32kL^2 < 0 \quad \text{maximum}
\]
so \( x = 0.578L \)
38. The electric power $P$ (in $W$) produced by a certain battery is given by $P = \frac{144r}{(r + 0.6)^2}$, where $r$ is the resistance in the circuit.

For what value of $r$ is the power a maximum?

i) The equation:

$$P = \frac{144r}{(r + 0.6)^2}$$

ii) Find the two derivatives of $P$.

$$\frac{dP}{dr} = -\frac{144(r - 0.6)}{(r + 0.6)^3}$$

$$\frac{d^2P}{dr^2} = \frac{288(r - 1.2)}{(r + 0.6)^4}$$

iii) Solve for the location of the minimum.

$$\frac{dP}{dr} = \frac{-144(r - 0.6)}{(r + 0.6)^3} = 0 \quad r = 0.6$$

Plug in $r$ into the second derivative to make sure the solution is for a maximum or just observe that if $r < 1.2$, $\frac{d^2P}{dr^2} = \frac{288(r - 1.2)}{(r + 0.6)^4}$ then is negative so the solution is at a maximum.

$$\frac{d^2P}{dr^2} = -\frac{288(0.6)}{(0.6 + 0.6)^4} < 0$$

maximum
Type 2 Max/Min

The second max/min type involves two equations.  \( u \) is a function of \( x \) and \( y \), and \( v \) is a function of \( x \) and \( y \). Find the max/min of one of them given the value of the other one.

14. The electric potential \( V \) on the line \( 3x + 2y = 6 \) is given by \( V = 3x^2 + 2y^2 \). At what point on this line is the potential a minimum?

i) The equations: \( V = 3x^2 + 2y^2 \) \( 6 = 3x + 2y \)

The second equation has the number so solve for \( x \) or \( y \) and plug it into the first equation.

\[
y = 3 - \frac{3}{2} x
\]

\[
V = 3x^2 + 2\left(3 - \frac{3}{2} x\right)^2 = 15x^2 - 18x + 18
\]

ii) Find the two derivatives of \( V \).

\[
\frac{dV}{dx} = 15x - 18
\]

\[
\frac{d^2V}{dx^2} = 15 > 0 \text{ minimum}
\]

iii) Solve for the location of the minimum.

\[
\frac{dV}{dx} = 15x - 18 = 0
\]

\[
x = \frac{18}{15} = 1.2 \quad y = 3 - \frac{3}{2}(1.2) = 1.2
\]

iv) Graph to check your answer.

Since the point of interest is \( x=1.2 \). What is \( y \)? Note this is the \( y \) value representing \( V \) for the graph, not the \( y \) of the problem.
Go back to the HOME screen to find $y$. The calculator knows the function $y_1$ since you just typed it in so perform the following calculation.

So we need YMAX at least 7.2. I will choose 15.

I will pick the following window.

Now graph then take the minimum (F5/3).
33. A box with a lid is to be made from a rectangular piece of cardboard 10 cm by 15 cm, as shown in Fig. 24.62. Two equal squares of side \(x\) are to be removed from one end, and two equal rectangles are to be removed from the other end so that the tabs can be folded to form the box with a lid. Find \(x\) such that the volume of the box is a maximum.

![Fig. 24.62](image)

**i) The equations.**

We need to create two formulas. To do this we need to label some more sides.

First note that \(u = v\) since the width of the box will be the same as the length of the lid.

The full width is \(W = x + v + x + u = 2x + 2v = 15\)

The full height is \(H = x + w + x = 2x + w = 10\)

The volume of the box is \(V = x \cdot w \cdot v\)

What do we actually need? Since \(V = x \cdot w \cdot v\), we need \(w\) in terms of \(x\) from \(2x + w = 10\) and \(v\) in terms of \(x\) from \(2x + 2v = 15\). The equations are then

\[
V = x \cdot w \cdot v \\
v = \frac{15}{2} - x \\
w = 10 - 2x
\]

Plugging into \(V\) we get

\[
V = x(10 - 2x)\left(\frac{15}{2} - x\right) = 2x^3 - 25x^2 + 75x
\]
where I used the TI expand function to get the product.

\[
\text{expand}\left(x \cdot (10 - 2 \cdot x) \cdot (15/2 - x)\right)
\]

\[
= 2 \cdot x^3 - 25 \cdot x^2 + 75 \cdot x
\]

ii) Find the two derivatives of V.

\[
\frac{dV}{dx} = 6x^2 - 50x + 75
\]

\[
\frac{d^2V}{dx^2} = 12x - 50
\]

iii) Solve for the location of the minimum.

\[
\frac{dV}{dx} = 6x^2 - 50x + 75 = 0
\]

Which one is it? Plug the answers into one of the equations above.

\[w = 10 - 2 \times 6.37 = -2.74\]

A dimension cannot be negative so \(x = 1.961\).

iv) Check the second derivative.

Is this a maximum or a minimum?

\[
\frac{d^2V}{dx^2} = 12(1.961) - 50 = -26.5 < 0
\]

It's a maximum.

The question was to find \(x\) so we are done.
i) The equations:
Height: \( d \) where \( d \) is the diameter of the semicircles
Perimeter: \( P = 2x + 2\pi r = 2x + \pi d = 400 \) where \( r = \frac{d}{2} \)
Area: \( A = xd + \pi r^2 \)

Solve the perimeter equation for \( d \):
\[
\pi d = 400 - 2x \\
d = \frac{400 - 2x}{\pi}
\]

Plug into the area equation of just the rectangle
\[
A = xd = x \left( \frac{400 - 2x}{\pi} \right) = \frac{2}{\pi} \left( 200x - x^2 \right)
\]

ii) Find the two derivatives of \( A \).
\[
\frac{dA}{dx} = \frac{2}{\pi} \left( 200 - 2x \right) = \frac{4}{\pi} \left( 100 - x \right)
\]
\[
\frac{d^2A}{dr^2} = -\frac{4}{\pi} < 0 \quad \text{maximum}
\]

iii) Solve for the location of the maximum.
\[
\frac{dA}{dx} = \frac{4}{\pi} \left( 100 - x \right) = 0
\]
\[
x = 100 \text{m}
\]

A race track 400 m long is to be built around an area that is a rectangle with a semicircle at each end. Find the open side of the rectangle if the area of the rectangle is to be a maximum. See Fig. 24.67.